Quantum from principles

Quantum theory

1) Pure states $\psi \in \mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements $(F_1, ..., F_n) \quad F_i \geq 0 \; \sum_i F_i = \mathbb{I}$

4) Probabilities $p(F_i|\psi) = \langle\psi|F_i|\psi\rangle$

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Introduction

- Quantum theory can be defined via a short number of clear postulates.
- Although these postulates are strictly speaking independent, one can ask questions of the following type:
- "Are there theories with the same pure states as quantum theory?"

Introduction

- Need a general framework to talk about theories.
- What type of theory is quantum theory?
- Is it an information theory? A logic? A form of mechanics? A general probability theory? A process theory?

Introduction

- Are there logics with the same measurement structure as quantum theory but different states?
- Are there operational theories with the same pure states and dynamics as quantum theory but different measurements?
- The answers to these questions will tell us how independent the postulate are.

Introduction

- Is it possible to change one part of quantum theory whilst keeping the others constant?
- Could we obtain a future theory of physics by making “perturbations” to the current theory? Or will we need a radical overhaul?

Introduction

- If we assume quantum theory is a form of logic, then the structure of measurements fully encodes the probability rule.
- Assuming it is a GPT means that the dynamical part fully determines the probabilistic part.
- Adopting different perspectives on quantum theory will allow for different insights into what it has to tell us.
- I don't argue that any of the perspectives I adopt today are the final word, rather they are useful stepping stones.

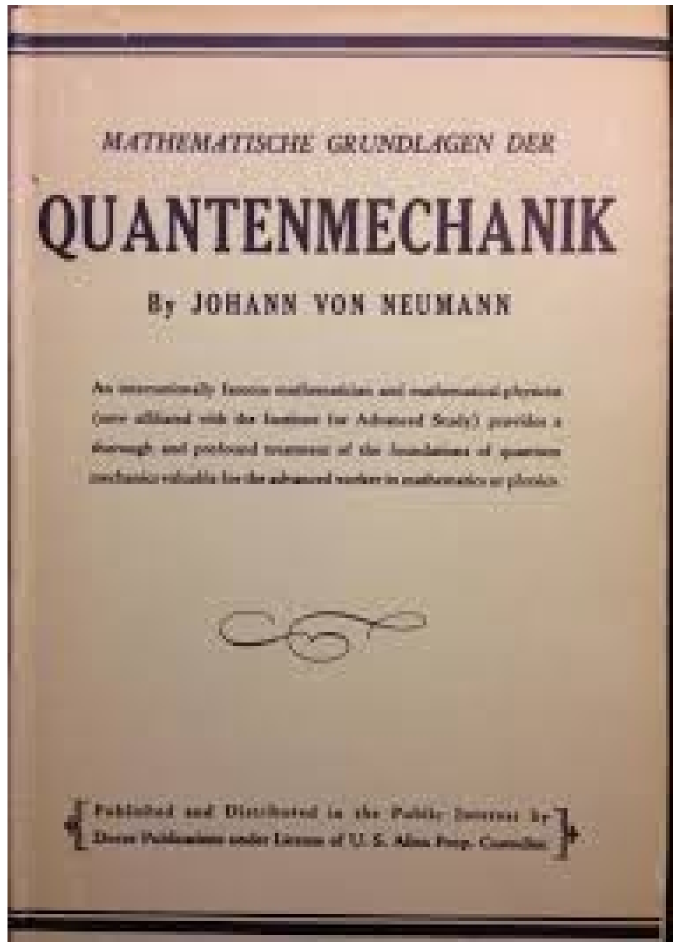
$p \implies q$

$L(\mathcal{H})$

$$\mu(\sum_i P_i) = \sum_i \mu(P_i)$$

Quantum logic and Gleason's theorem

Mathematical Foundations of Quantum Mechanics, Von Neumann (1932)



Annals of Mathematics
Vol. 37, No. 4, October, 1936

THE LOGIC OF QUANTUM MECHANICS

By Garrett Birkhoff and John von Neumann

(Received April 4, 1936)

Quantum logic

- Facts about quantum systems do not obey the rules of classical logic
- Need a new set of rules (a quantum logic) to reason about propositions

Classical logic

Classical logic (propositional)

- A set of rules about propositions
- Propositions can be true or false
- Can connect propositions together to form formulae using connectives: “and” , “or”
- One connective is implication $\implies$

Classical logic (propositional)

- A set of rules about propositions
- Propositions can be true or false

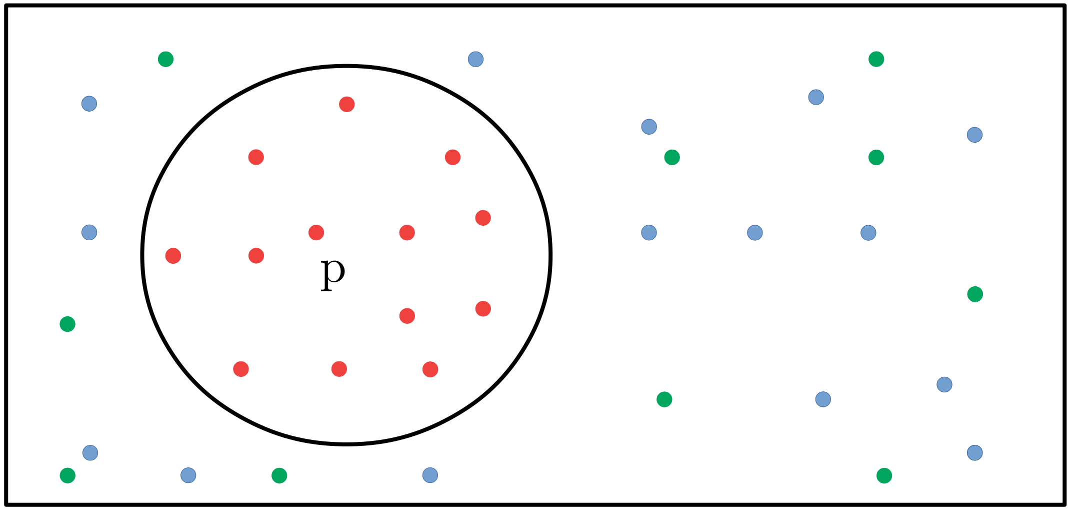
P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

Classical logic

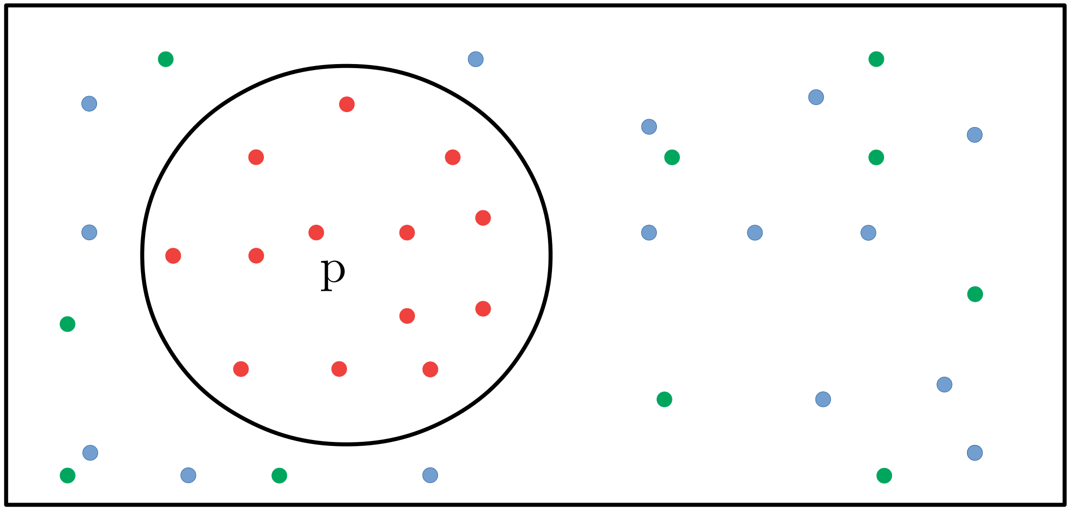
- A set of rules about propositions
- Propositions can be true or false

$$(p \text{ and } q) \implies p$$

Classical logic as set theory

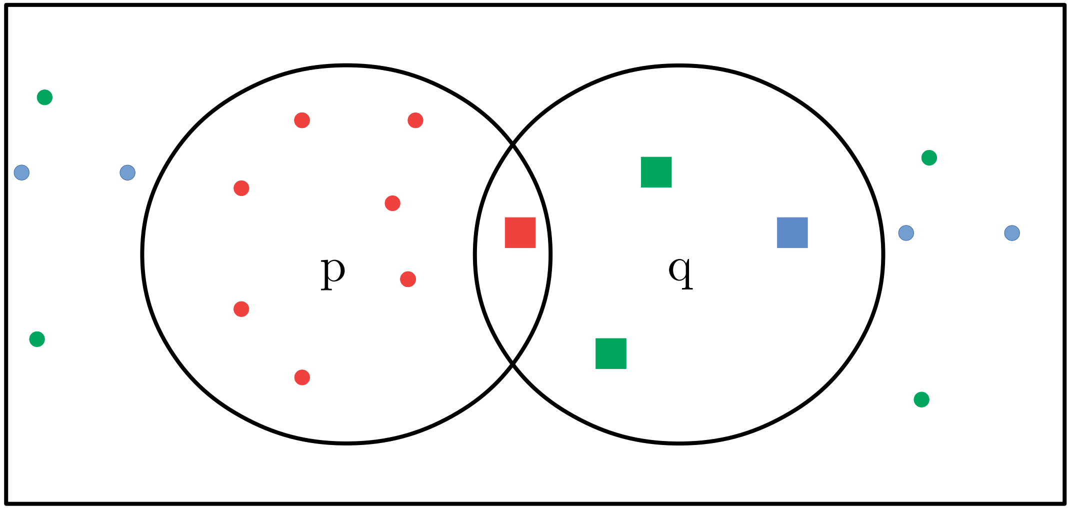


Classical logic as set theory



p = being red = all red objects

Logical connectives in set theoretic language

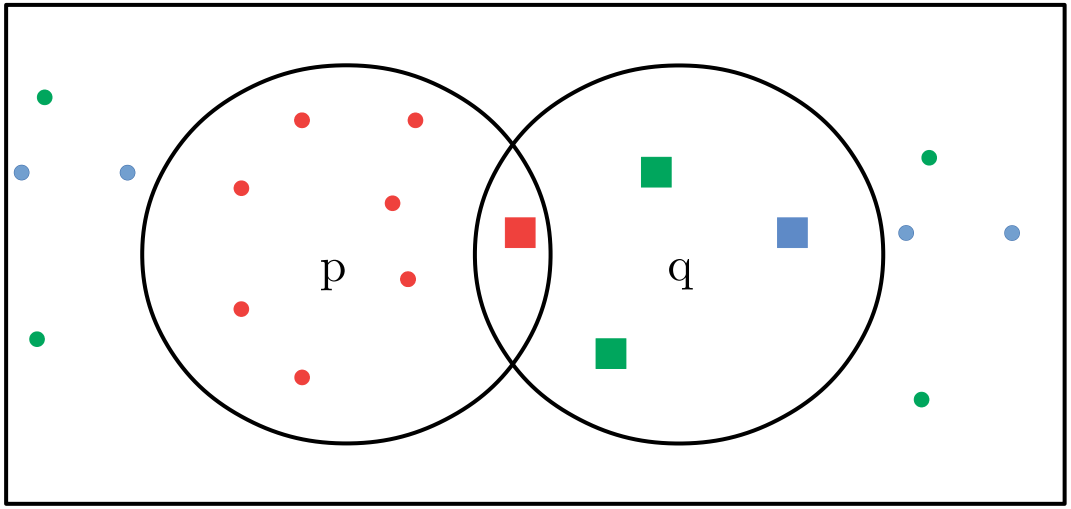


p = being red = all red objects

q = being square = all square objects

Logical connectives in set theoretic language

- “and” is set intersection



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Logical connectives in set theoretic language

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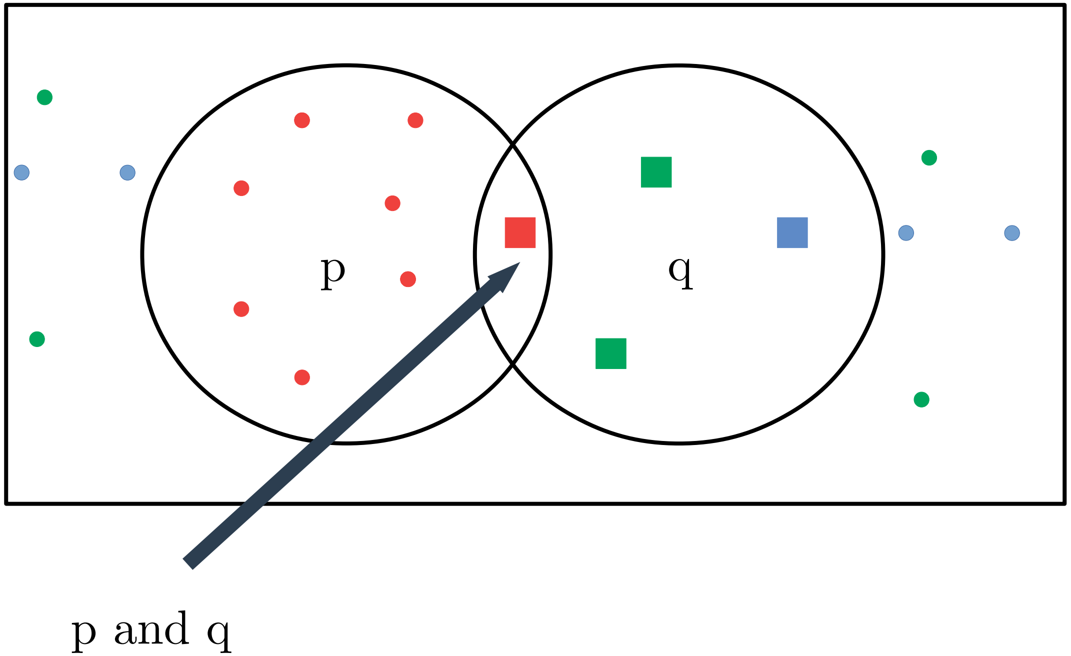
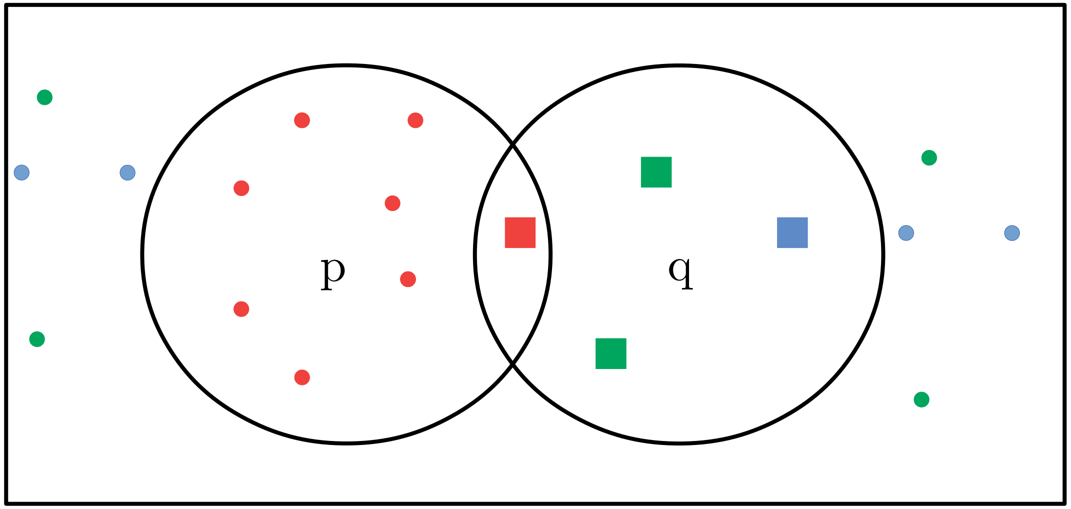


p = being red = all red objects

q = being square = all square objects

Logical connectives in set theoretic language

- "or" is set union



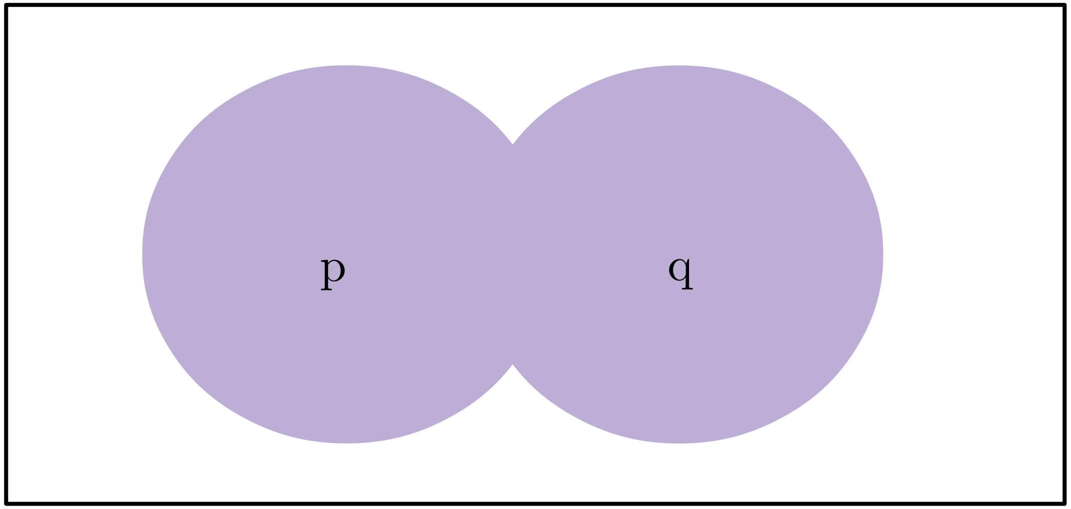
p or q

p = being red = all red objects

q = being square = all square objects

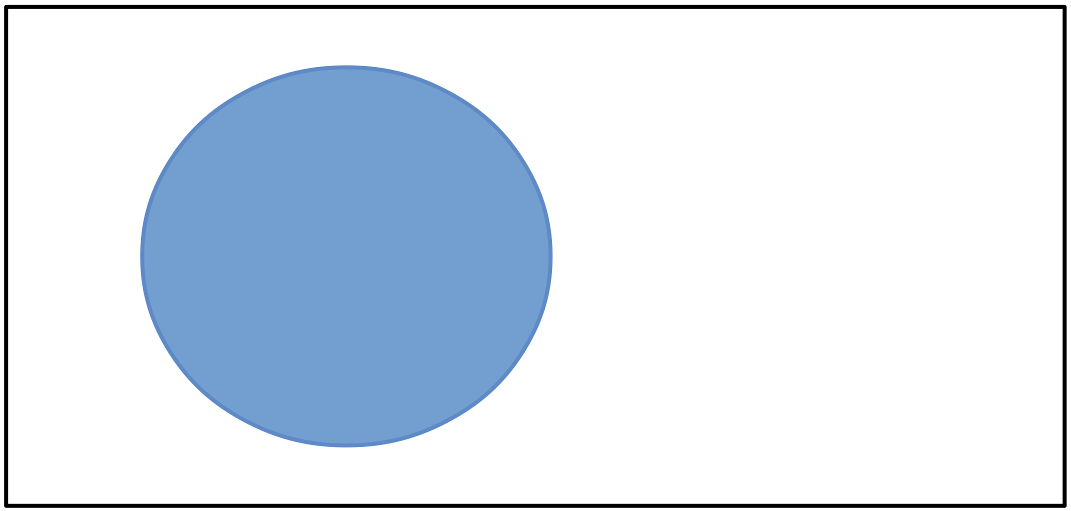
Logical connectives in set theoretic language

- "or" is set union



p or q

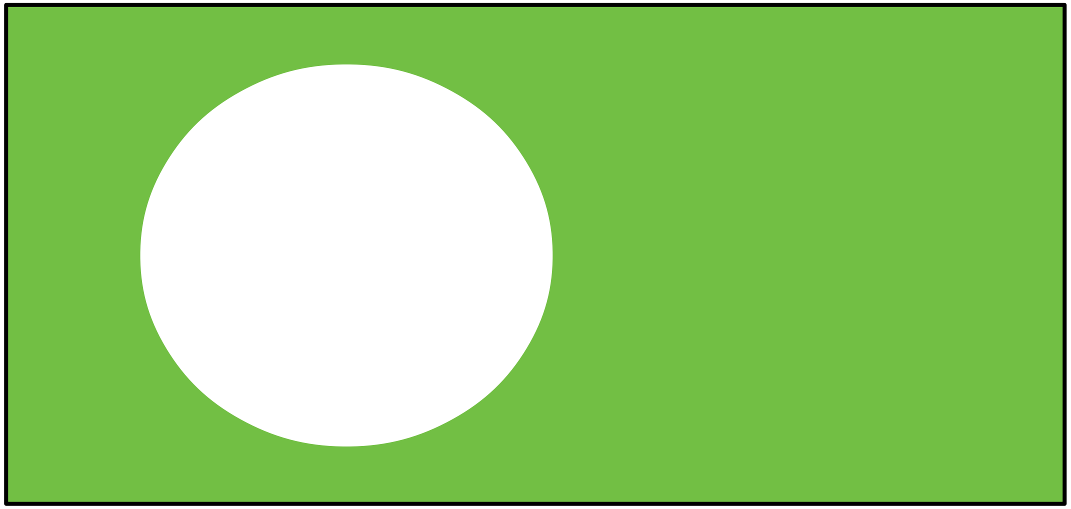
Classical logic as set theory



p = "particle is in the blue circle"

Logical connectives in set theoretic language

- “not” is complement set

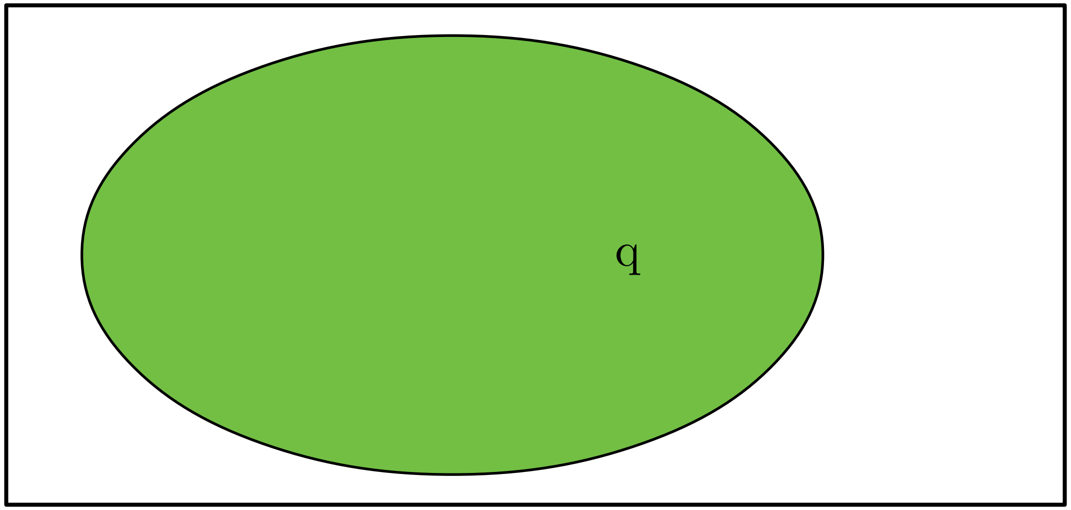


not p = “particle is not in the blue circle”

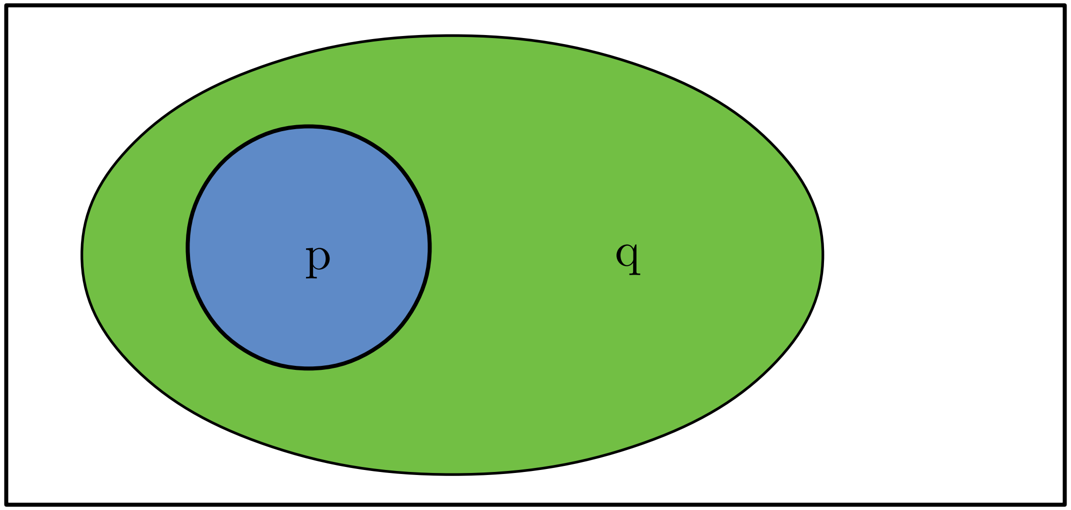
Logical connectives in set theoretic language

- What is implication?
- All things p are also q $p \implies q$
- "Being a square implies having four sides"
- All squares are four sided

Implication



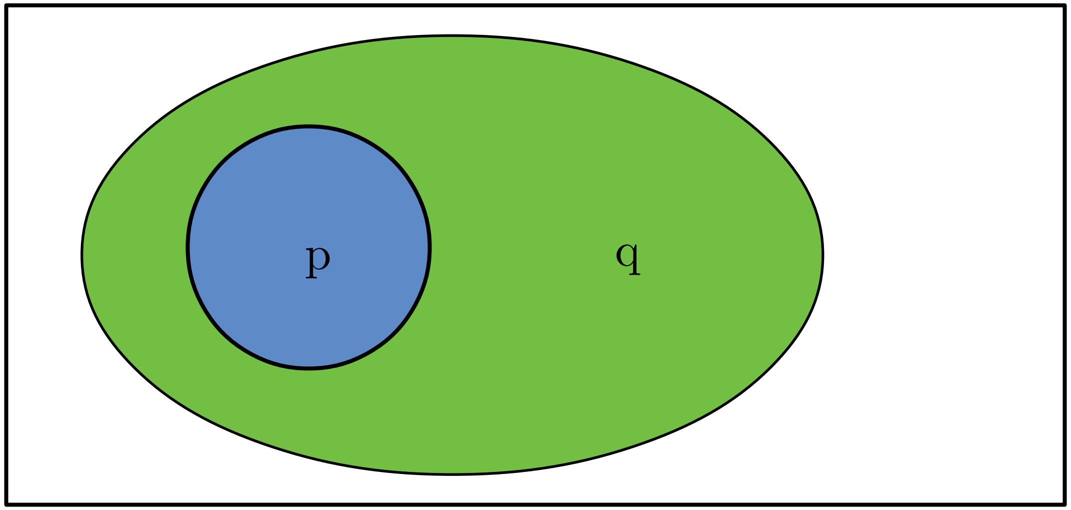
Implication



Implication as set inclusion

- Implication is given by set inclusion

$$p \subseteq q$$



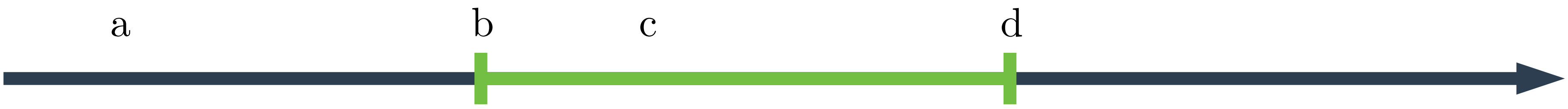
Classical logic

- Classical physics:

p = “particle is in interval [a,c]”

Classical logic

- Classical physics:



q = "particle is in interval [b,d]"

Classical logic

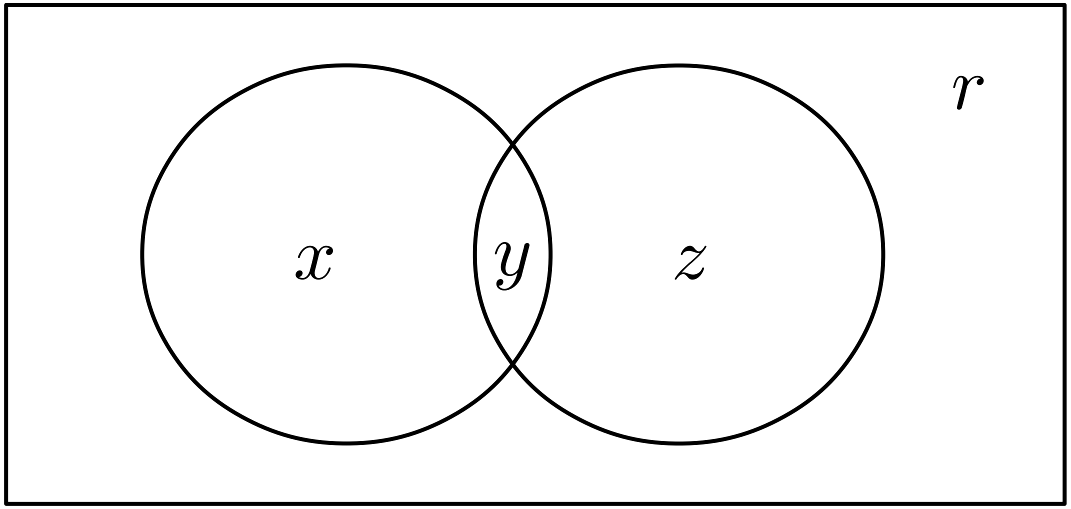
- Classical physics:

p and q = “particle is in interval [b,c]”

A simple example

$p = \{x, y\}$

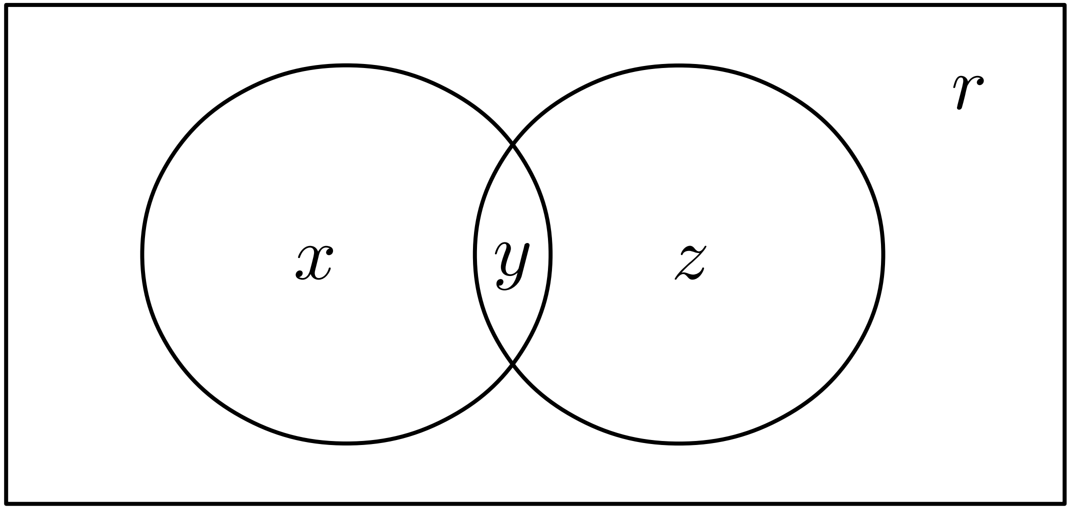
$q = \{y, z\}$



p and $q = \{y\}$

A simple example

$p = \{x, y\}$ $q = \{y, z\}$



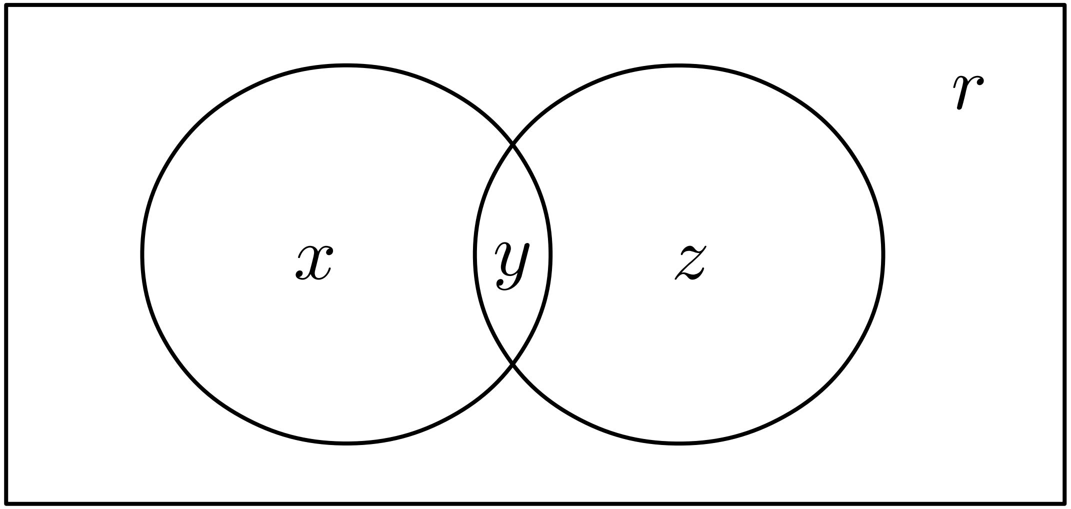
$$p \text{ and } q = \{y\}$$

$$\{y\} \subset \{x, y\} \text{ equivalent to } (p \text{ and } q) \implies p$$

A simple example

$p = \{x, y\}$

$q = \{y, z\}$



$p \text{ or } q = \{x, y, z\}$

$\neg p = \{z, r\}$

$\neg q = \{x, r\}$

Logic as set theory

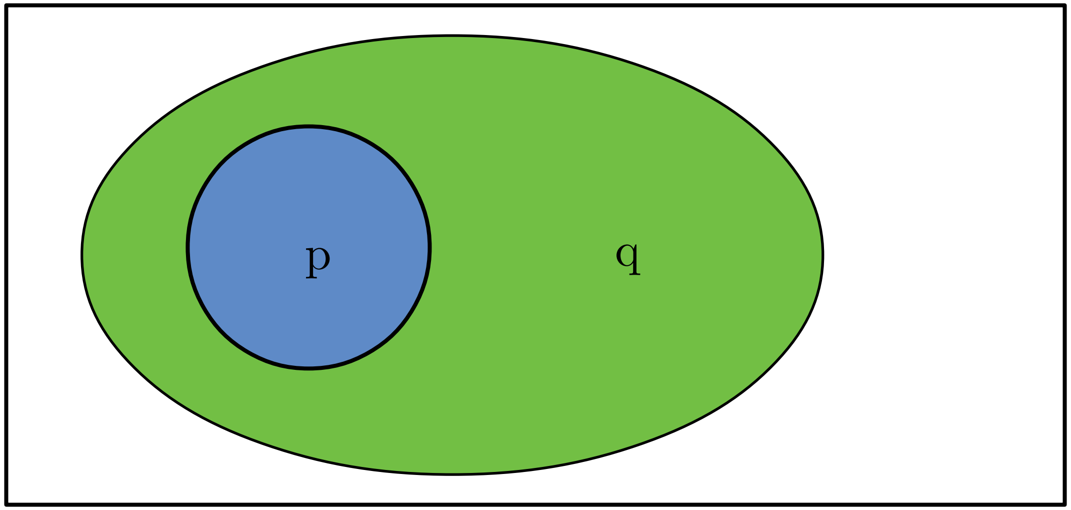
- Propositions correspond to sets.
- Implication amongst propositions given by set inclusion.
- Set inclusion is a partial order:

$$\{x\} \subseteq \{x, y\}$$

$$\{x\} \quad \{y, z\}$$

Implication

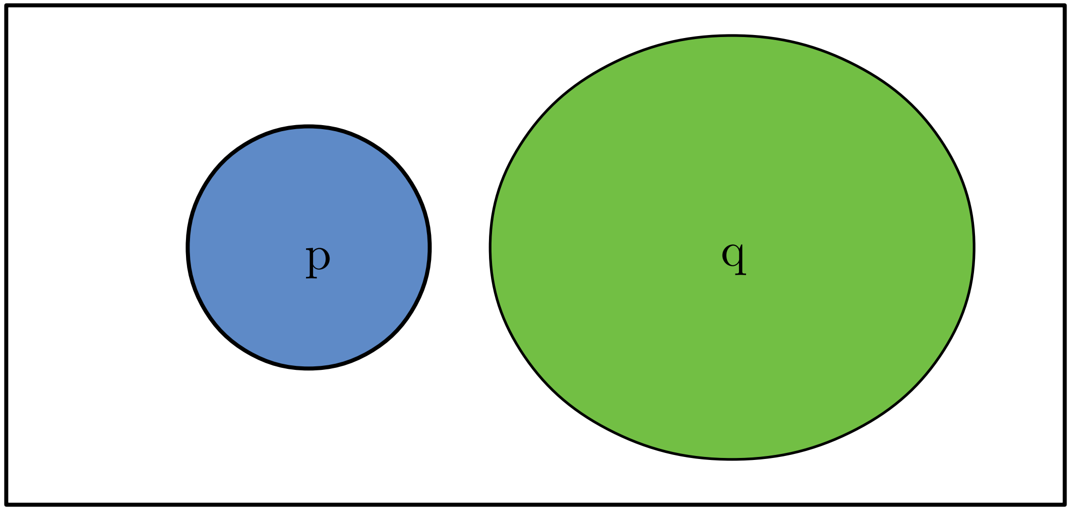
$$p \subseteq q$$



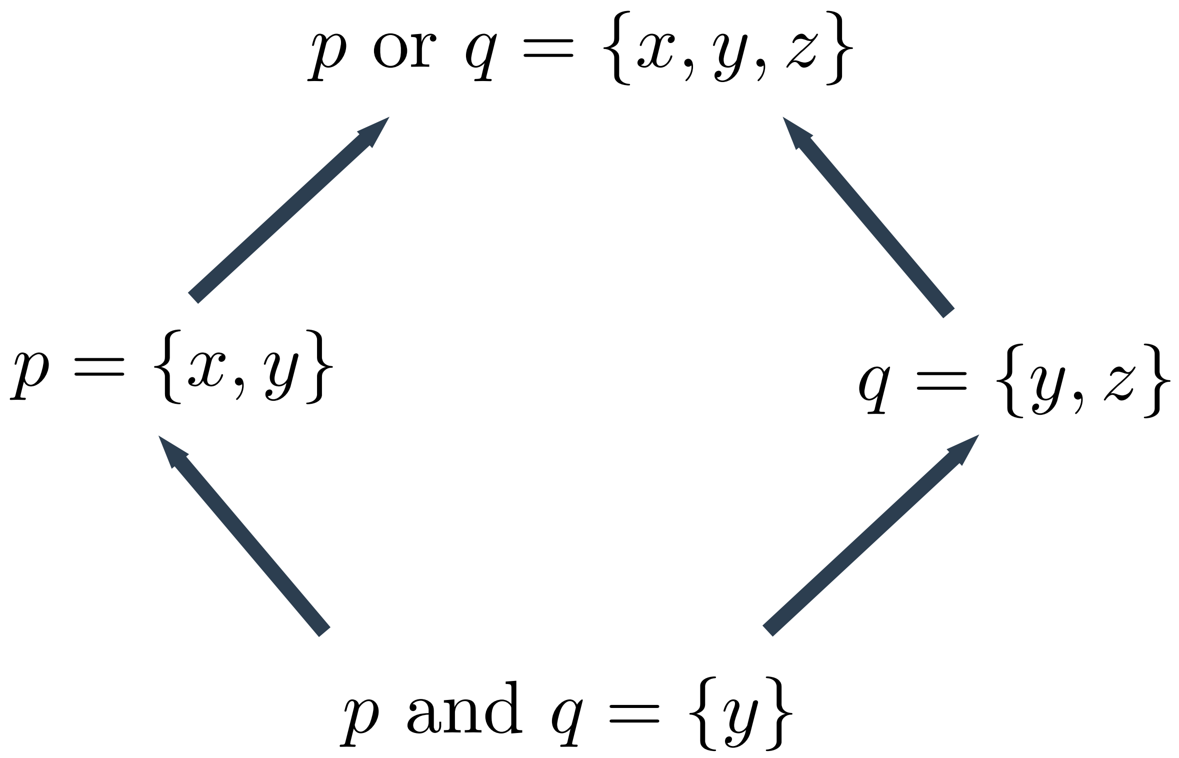
Implication

$$p \not\subseteq q$$

$$q \not\subseteq p$$



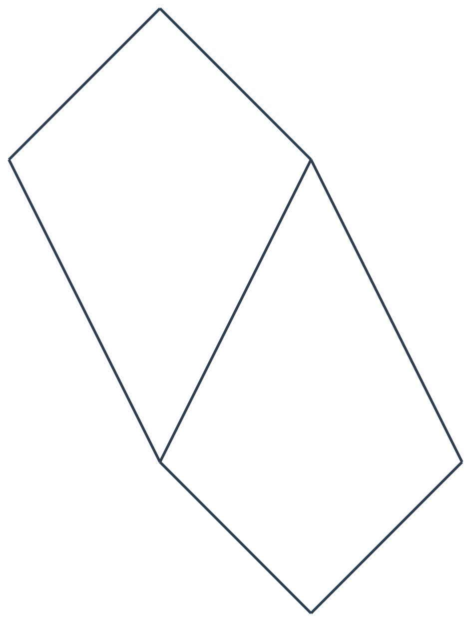
Lattice

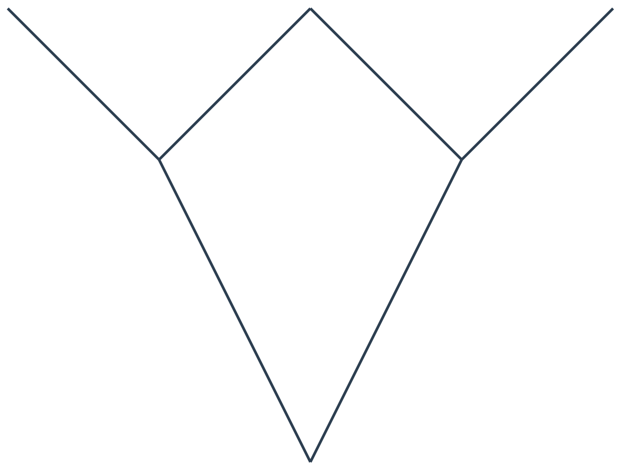


Complete lattice

- Complete lattice: Partially ordered set where every pair of elements has a “and” and “or” defined (meet and join)
- Meet: greatest lower bound
- Join: least upper bound

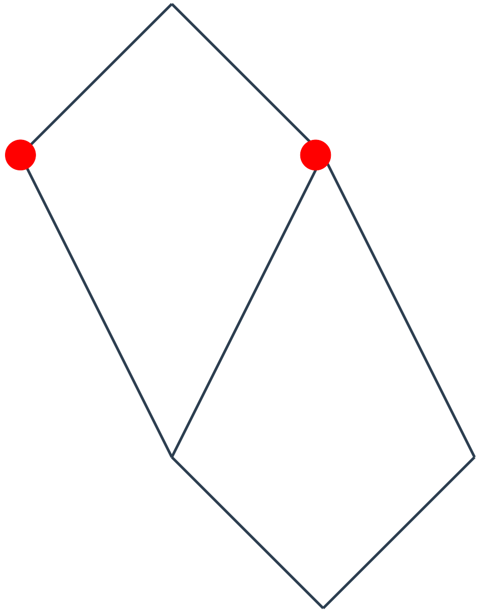
Complete lattices

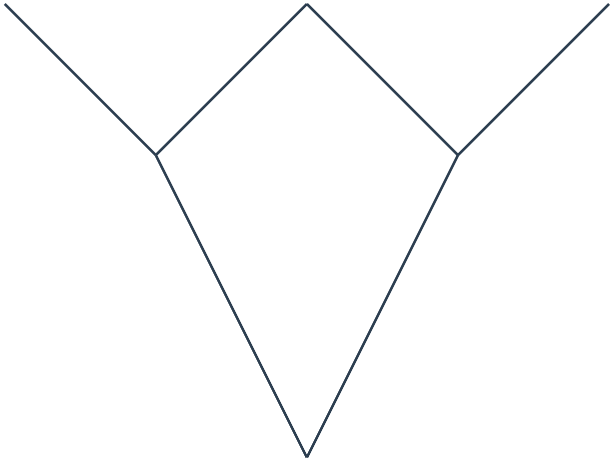




Complete lattices

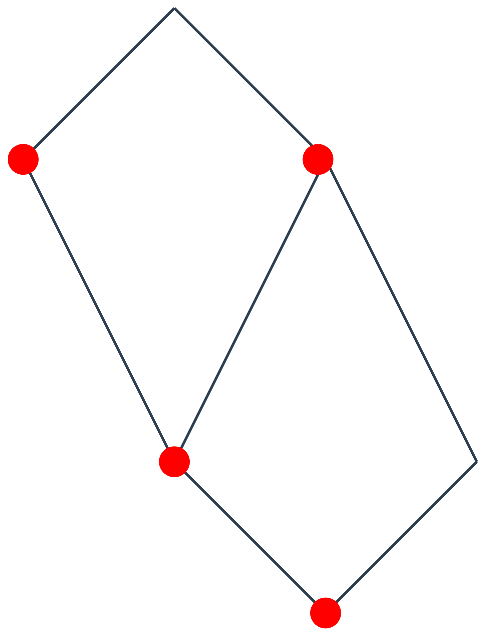
Meet? Greatest lower bound

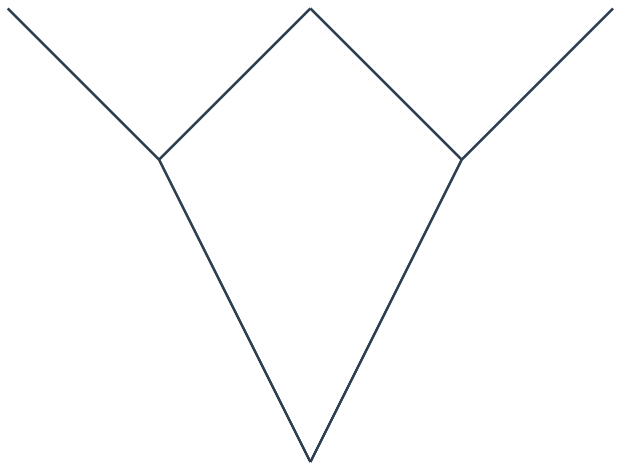




Complete lattices

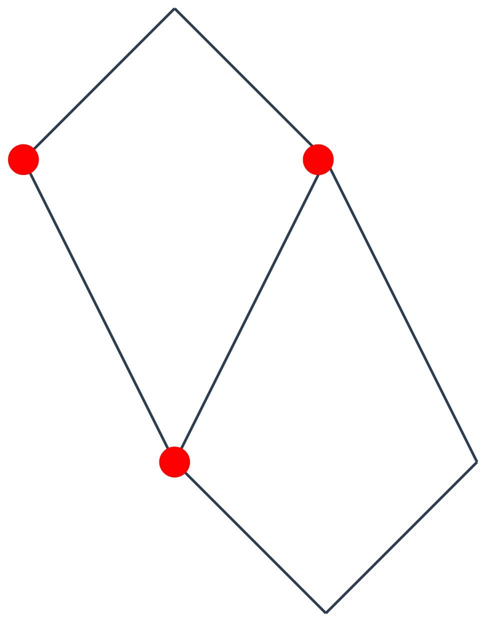
Meet? Greatest lower bound

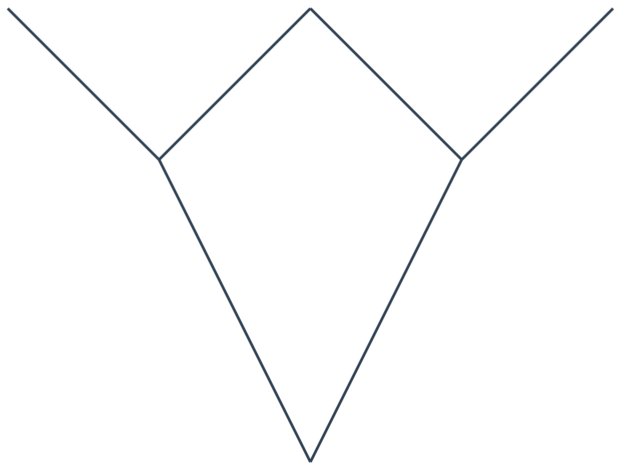




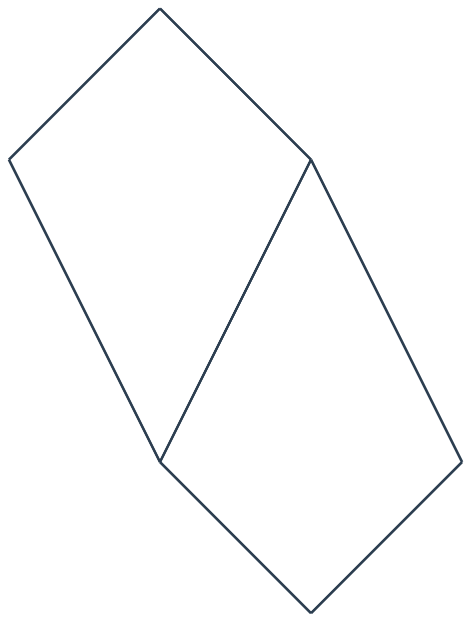
Complete lattices

Meet? Greatest lower bound

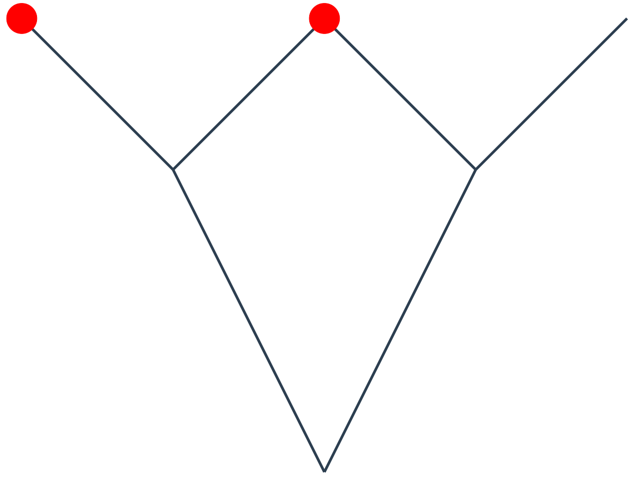




Complete lattices



Join? Lowest upper bound



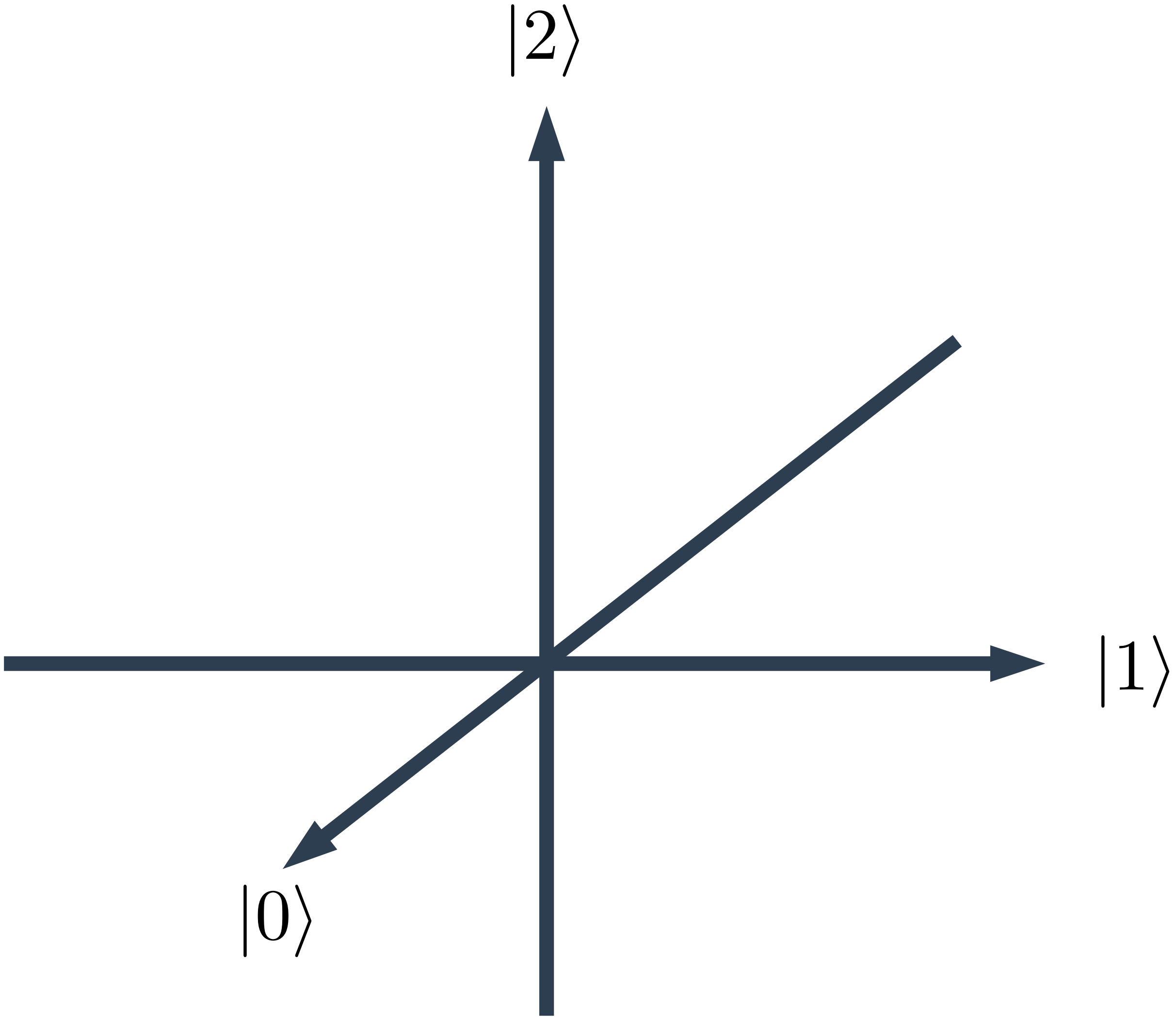
- A logic is just a complete lattice
- Classical logic is distributive lattice
- p and (q or r) = (p and q) or (p and r)
- p or (q and r) = (p or q) and (p or r)

- A logic is just a complete lattice
- Classical logic is distributive lattice
- Quantum logic is a non-distributive lattice
- We have a vantage point to talk about general non-classical logics, of which quantum logic is an example.

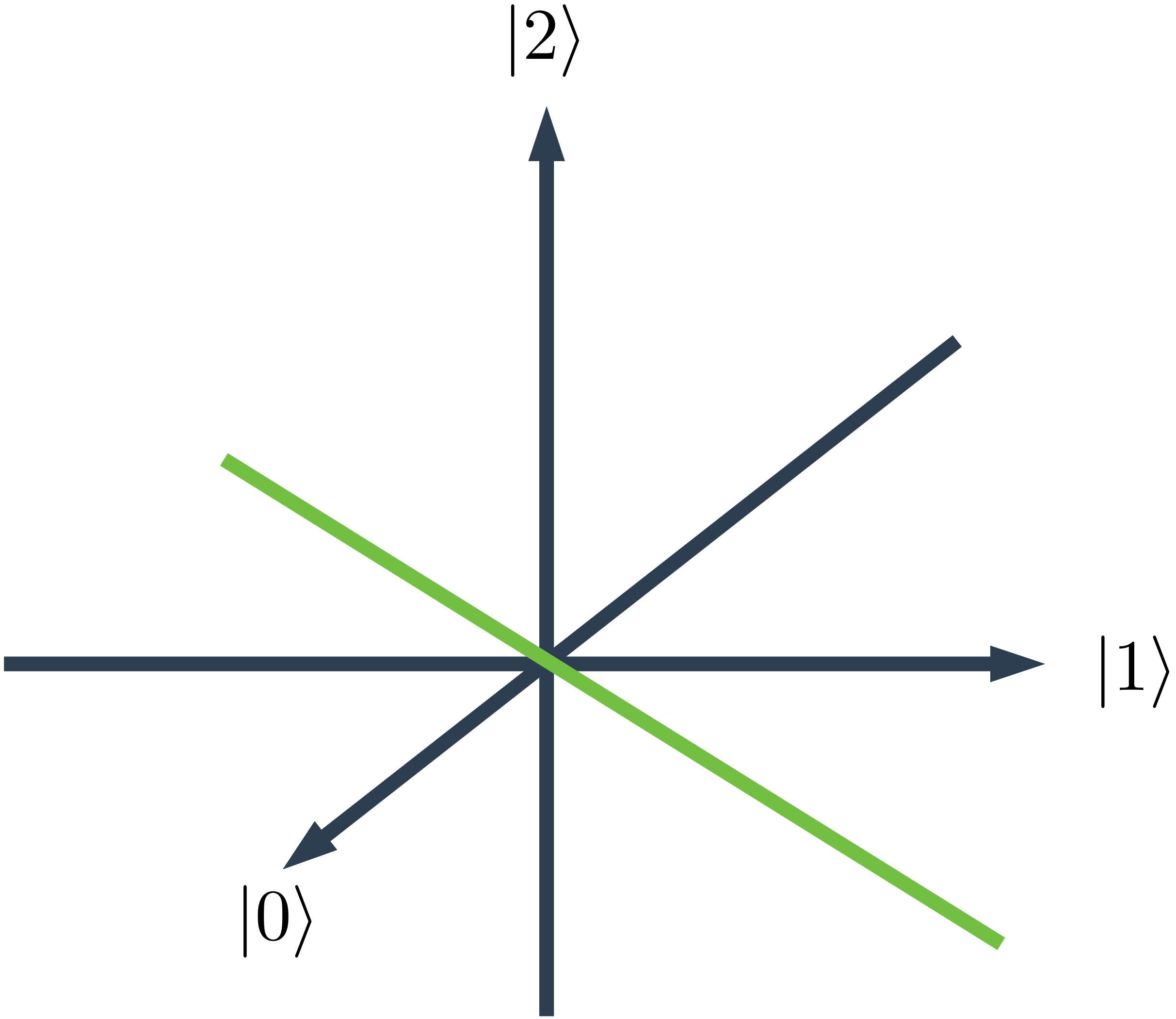
Quantum logic

- Propositions correspond to subspaces of H.
- Examples: $|0\rangle$, $|+\rangle$, $\text{span}(|0\rangle, |1\rangle)$
- Quantum logic is a lattice of subspaces

Quantum logic

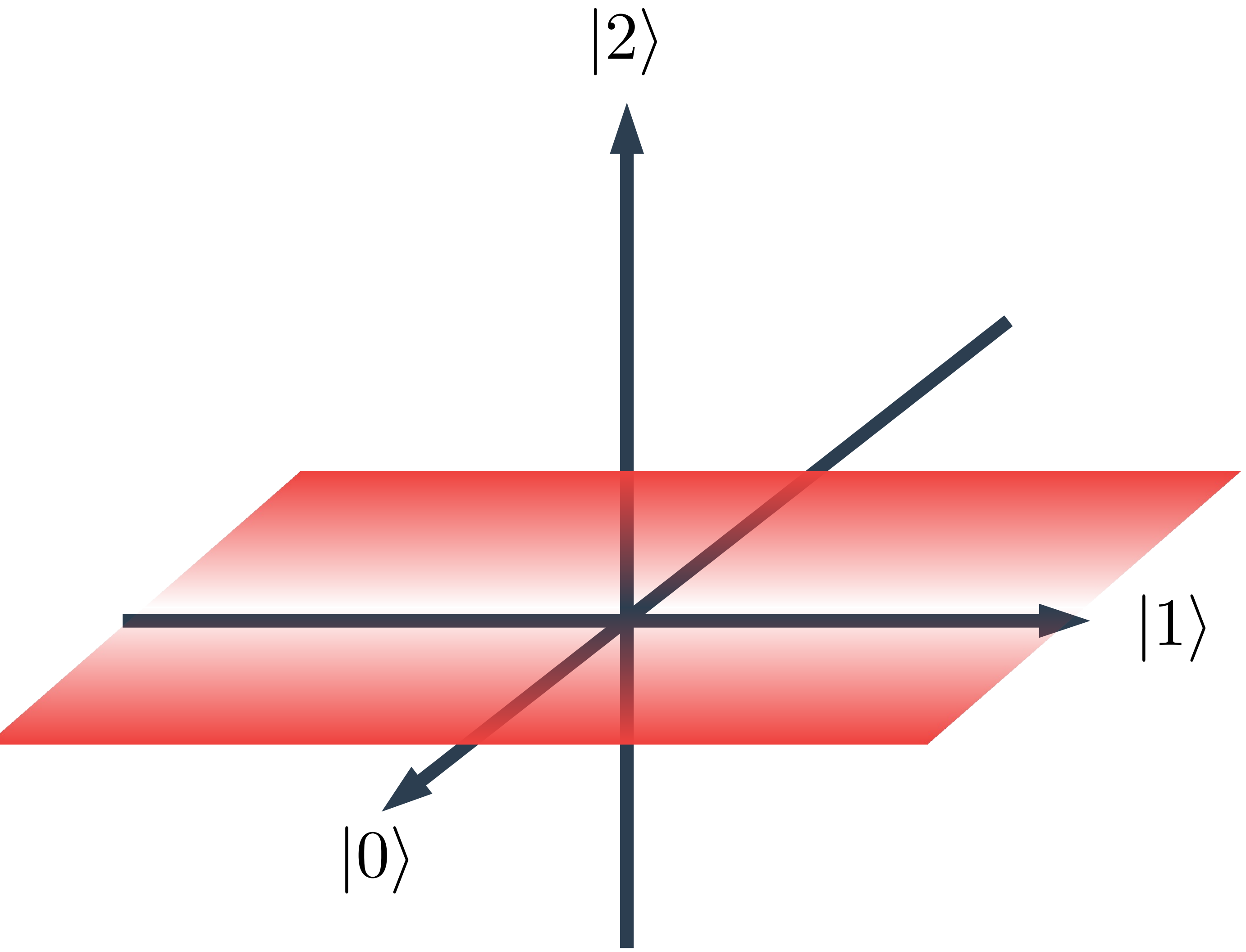


Quantum logic



$p = |+\rangle$

Quantum logic

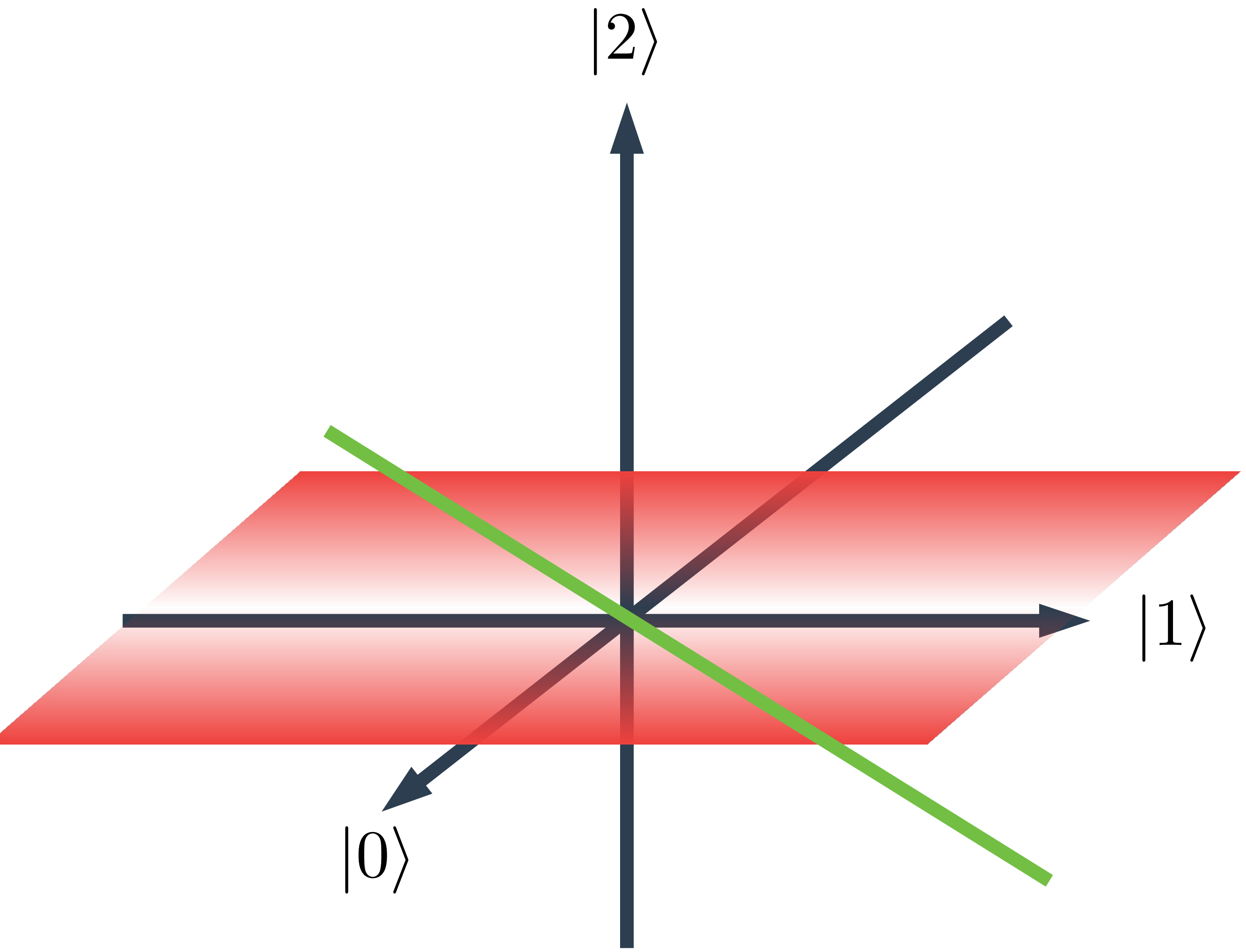


$$q = \mathrm{span}(|0\rangle, |1\rangle)$$

Quantum logic

- Propositions correspond to subspaces of H.
- Implication given by inclusion of subspaces

Quantum logic



$$p = |+\rangle$$

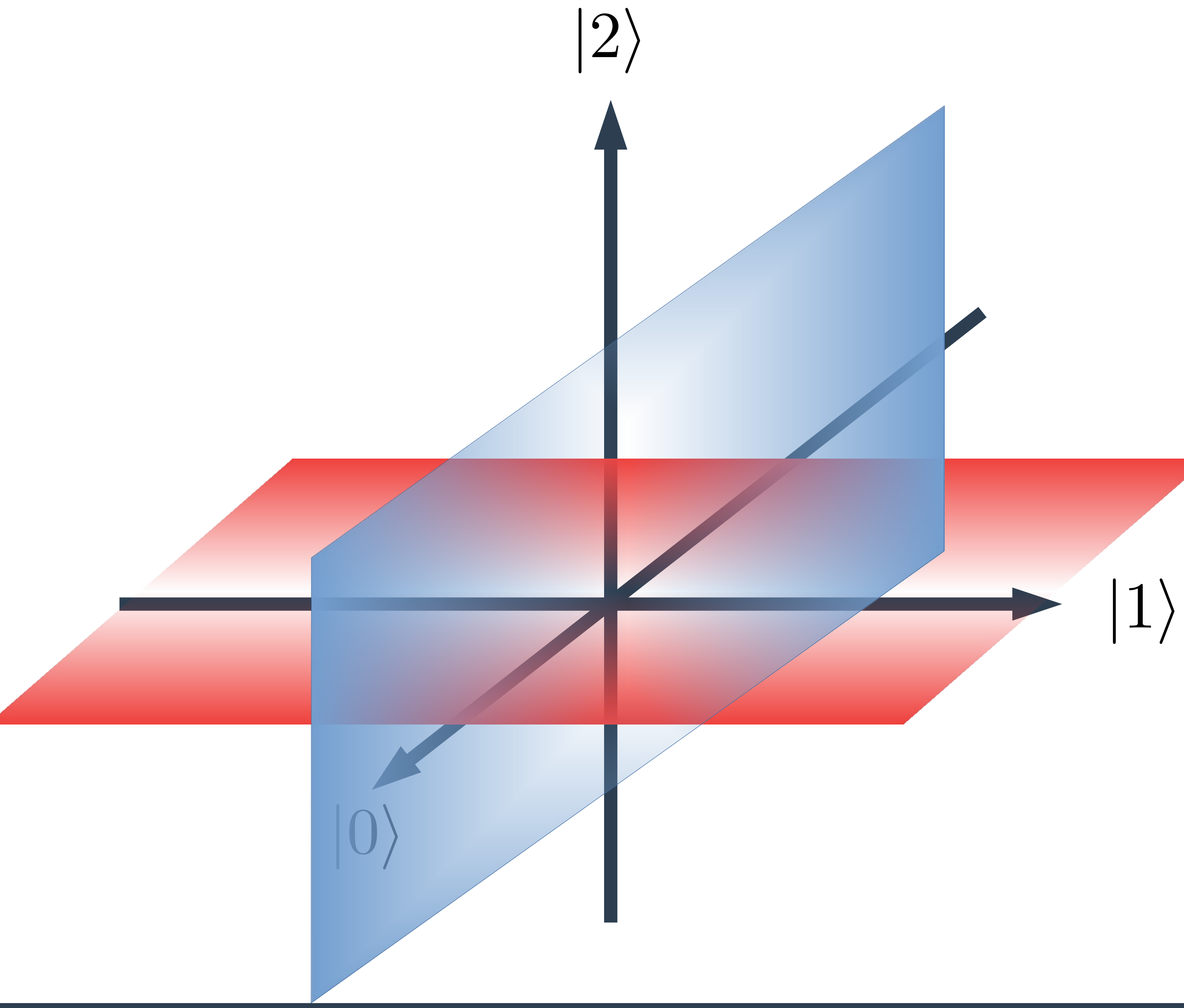
$$q = \text{span}(|0\rangle, |1\rangle)$$

$$p \implies q$$

Quantum logic

- Propositions correspond to subspaces of H.
- “and” is given by intersection

Quantum logic



$$p = \mathrm{span}(|0\rangle, |2\rangle)$$

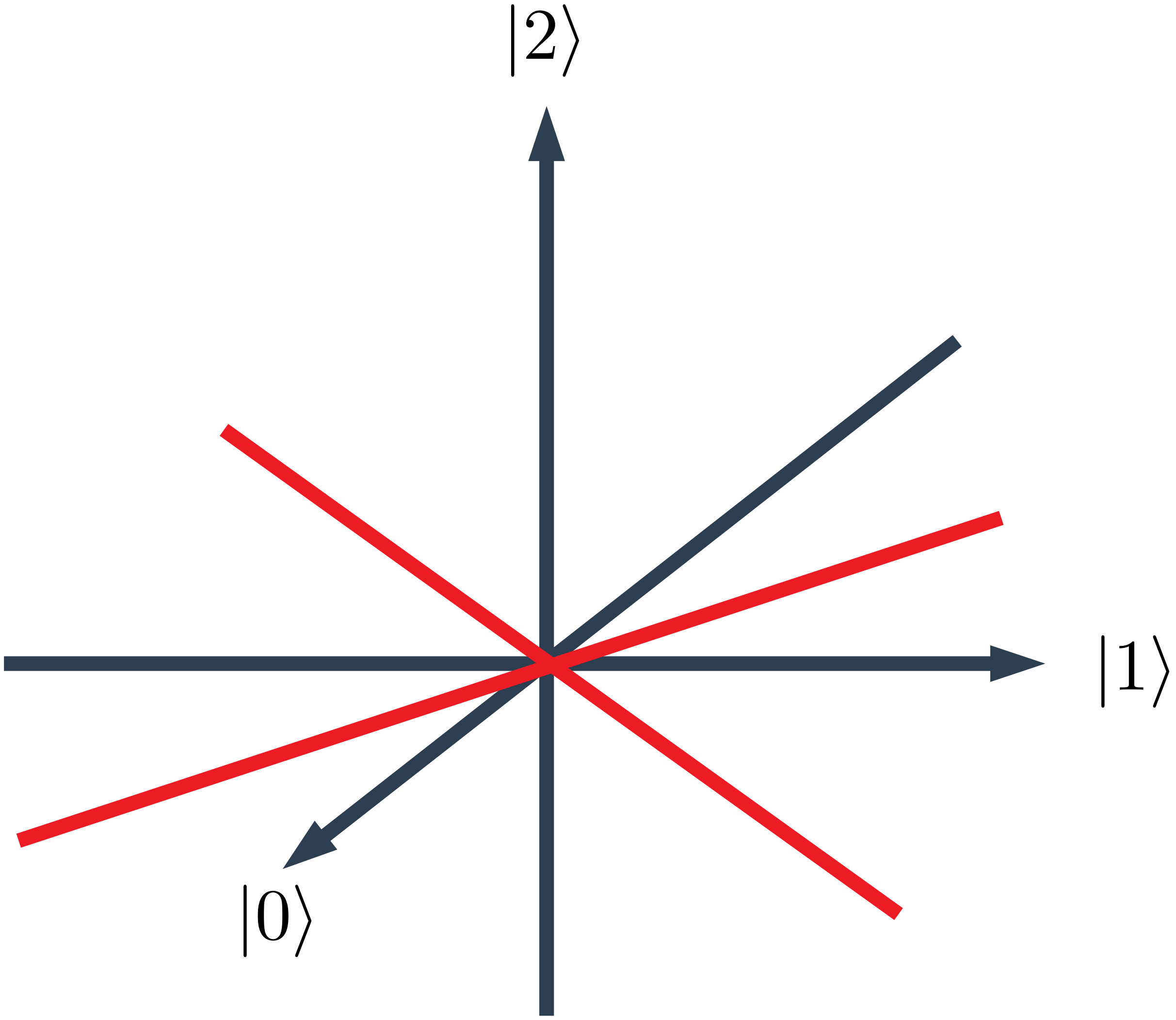
$$q = \mathrm{span}(|0\rangle, |1\rangle)$$

$$p \text{ and } q = |0\rangle$$

Quantum logic

- Propositions correspond to subspaces of H.
- “or” is given by span

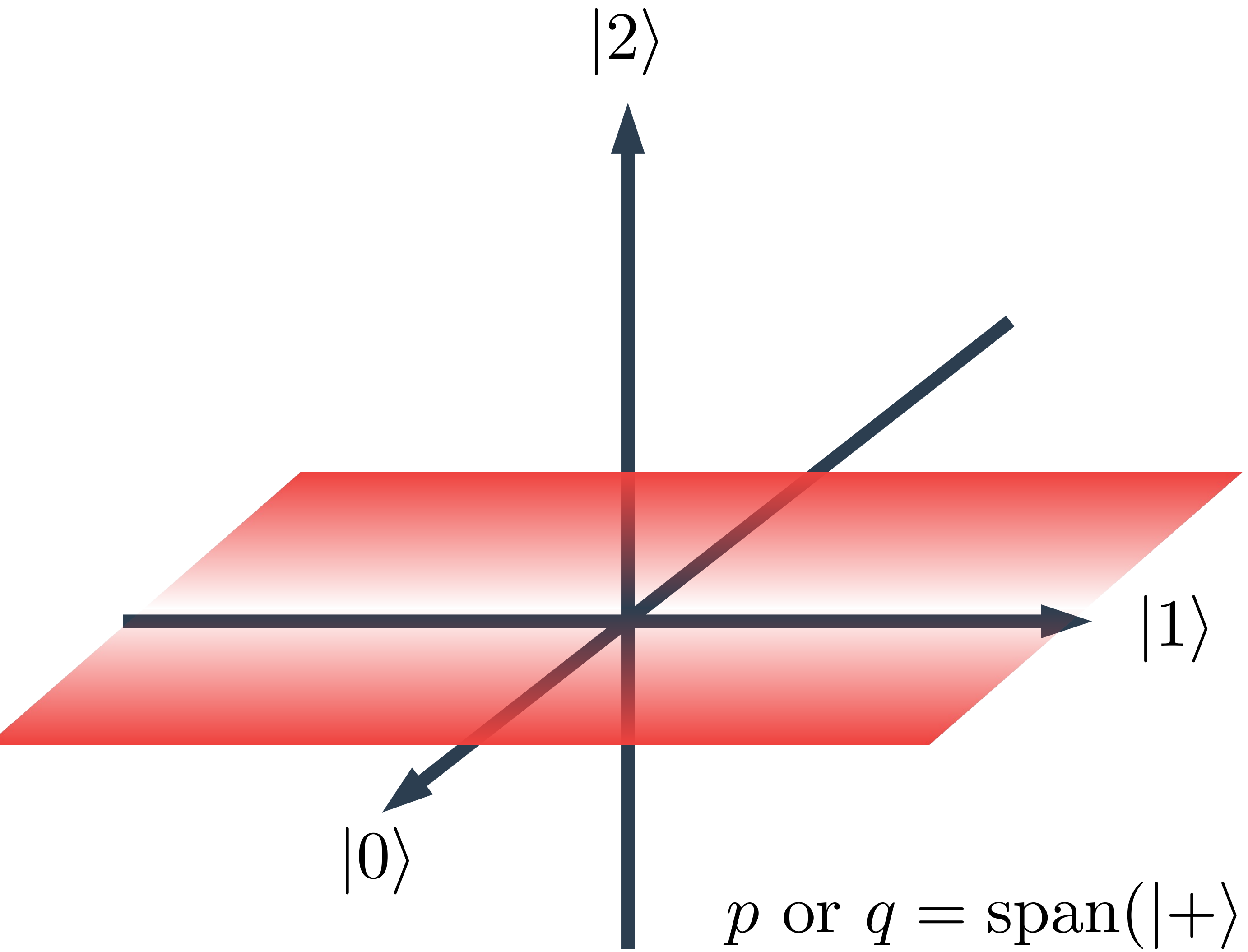
Quantum logic



$$p = |+\rangle$$

$$q = |-\rangle$$

Quantum logic



$$p = |+\rangle$$

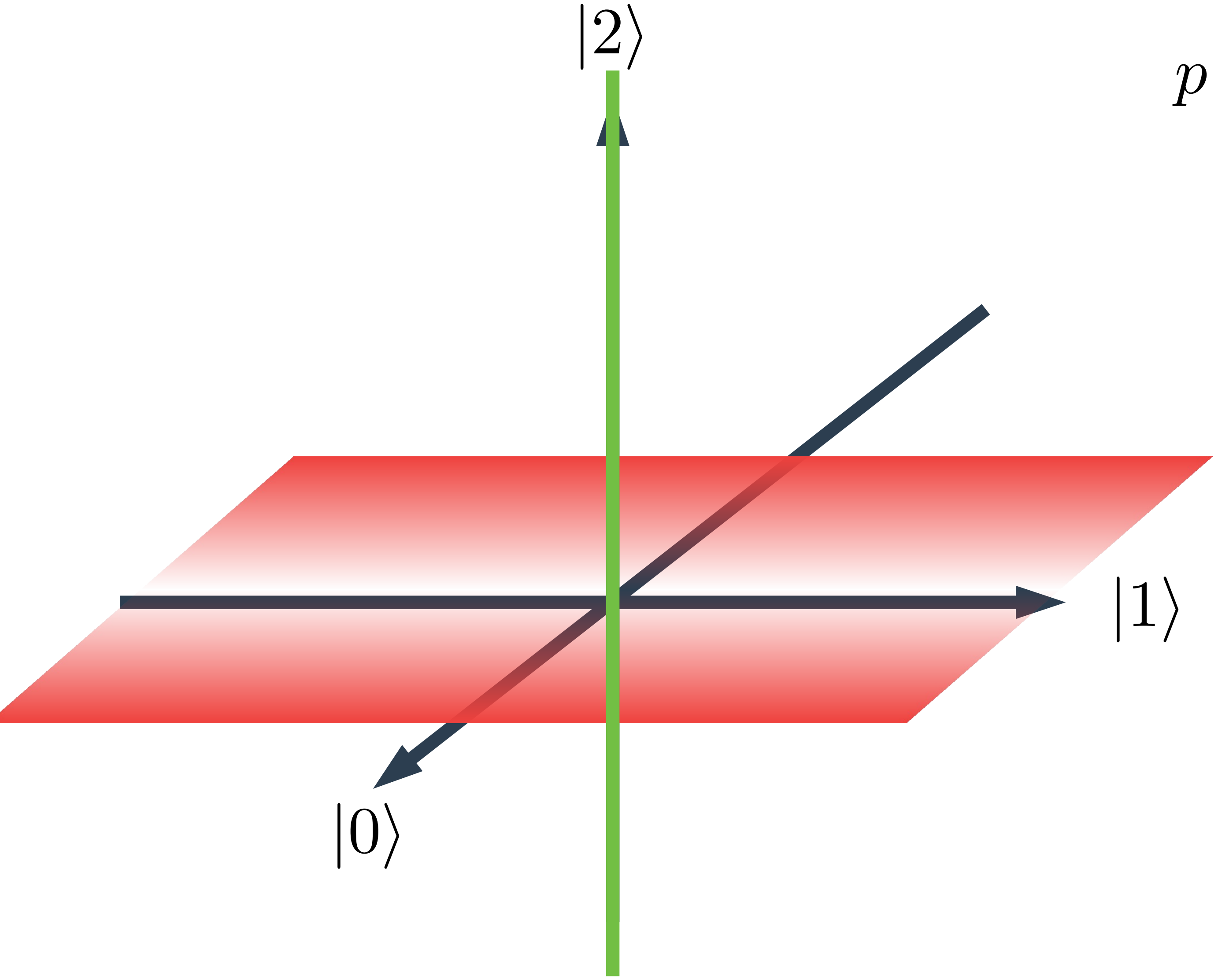
$$q = |-\rangle$$

$$p \text{ or } q = \operatorname{span}(|+\rangle, |-\rangle) = \operatorname{span}(|0\rangle, |1\rangle)$$

Quantum logic

- Propositions correspond to subspaces of H.
- Negation is given by orthogonal subspace

Quantum logic



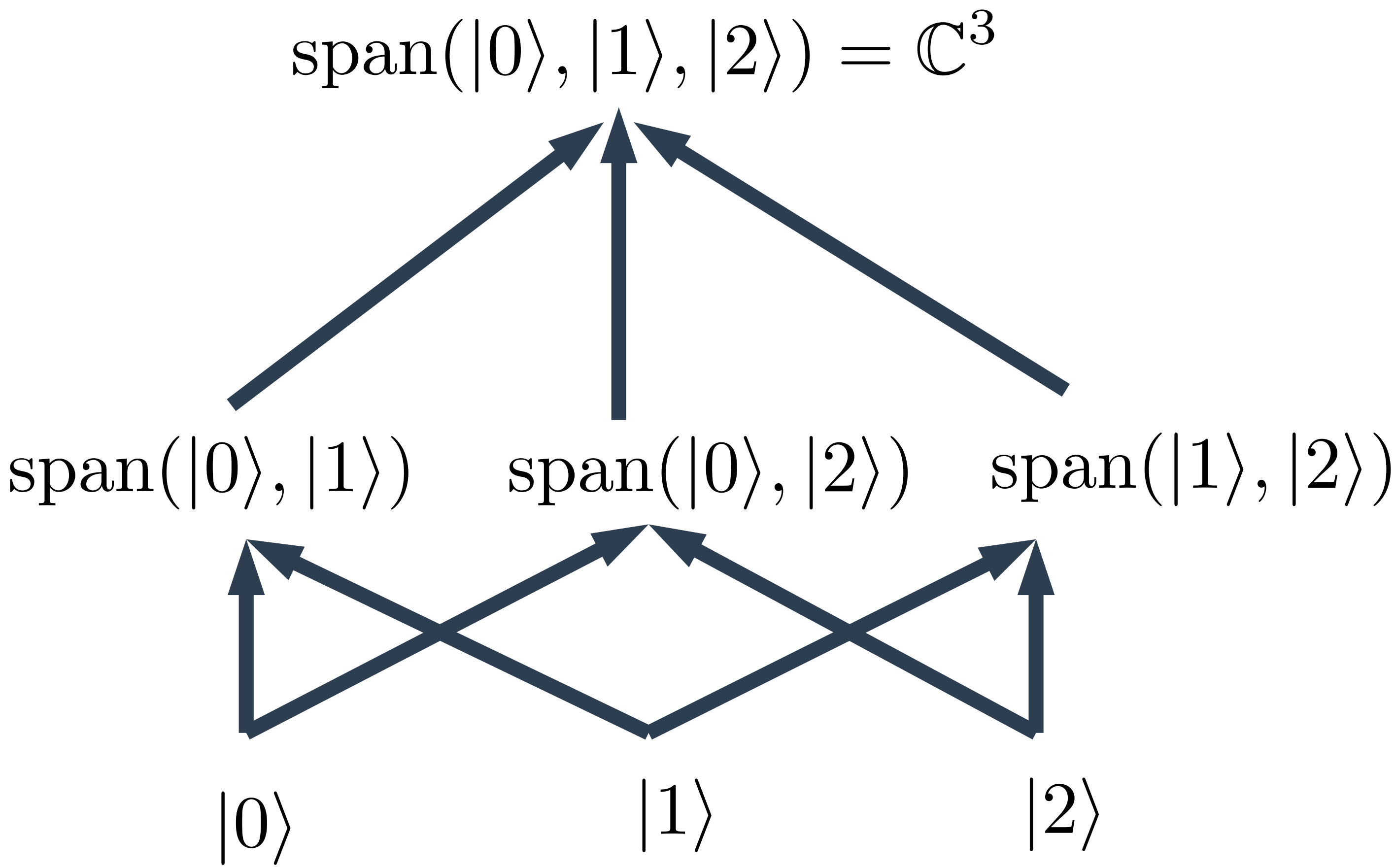
$$p = \mathrm{span}(|0\rangle, |1\rangle)$$

$$\neg p = |2\rangle$$

Quantum logic

- Propositions correspond to subspaces of H.
- Implication given by inclusion of subspaces
- “and” is given by intersection
- “or” is given by span
- Negation is given by orthogonal complement (orthocomplemented lattice)

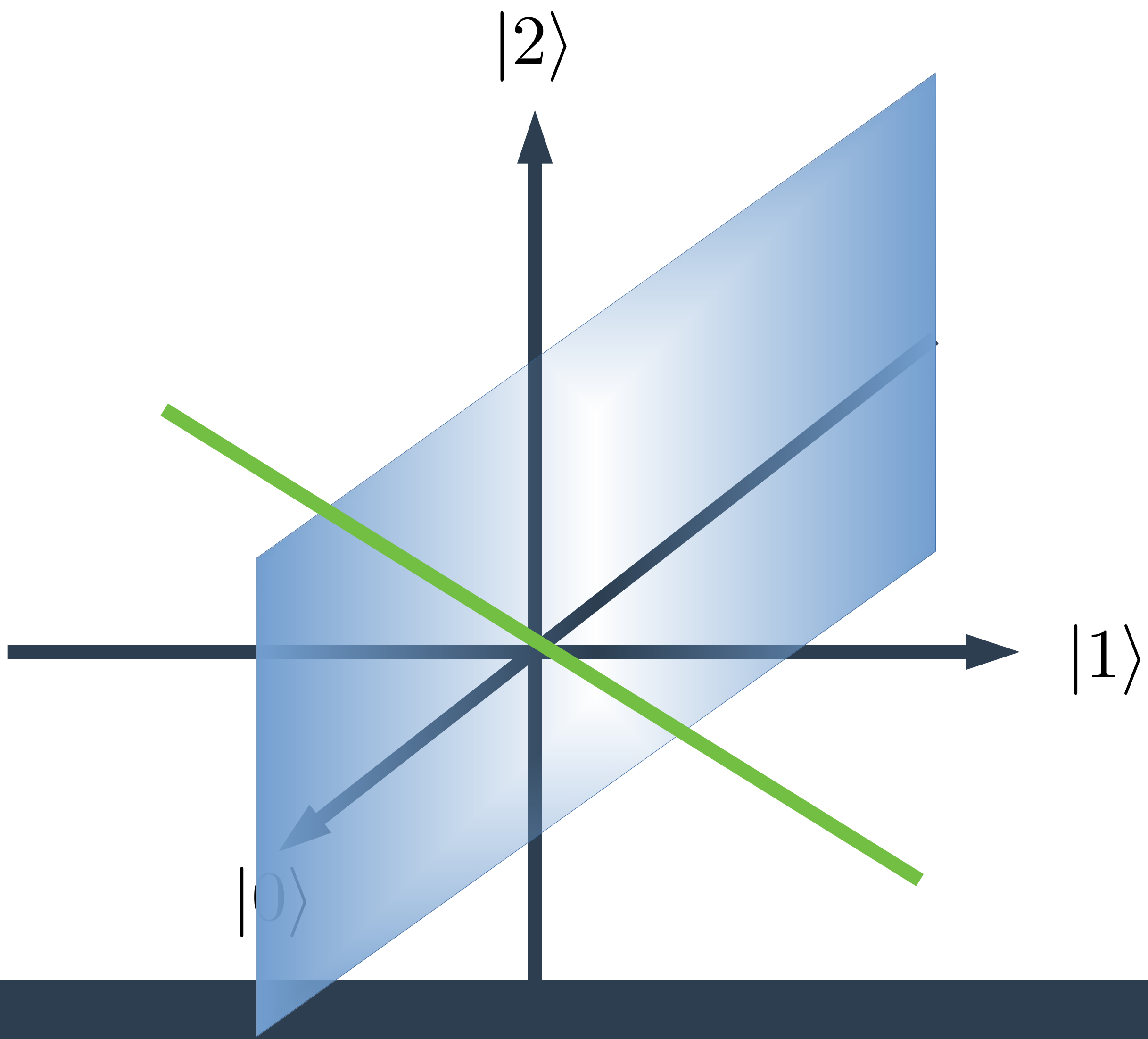
Quantum logic



Non-distributivity of quantum logic

$$p \text{ and } (q \text{ or } r) \neq (p \text{ and } q) \text{ or } (p \text{ and } r)$$

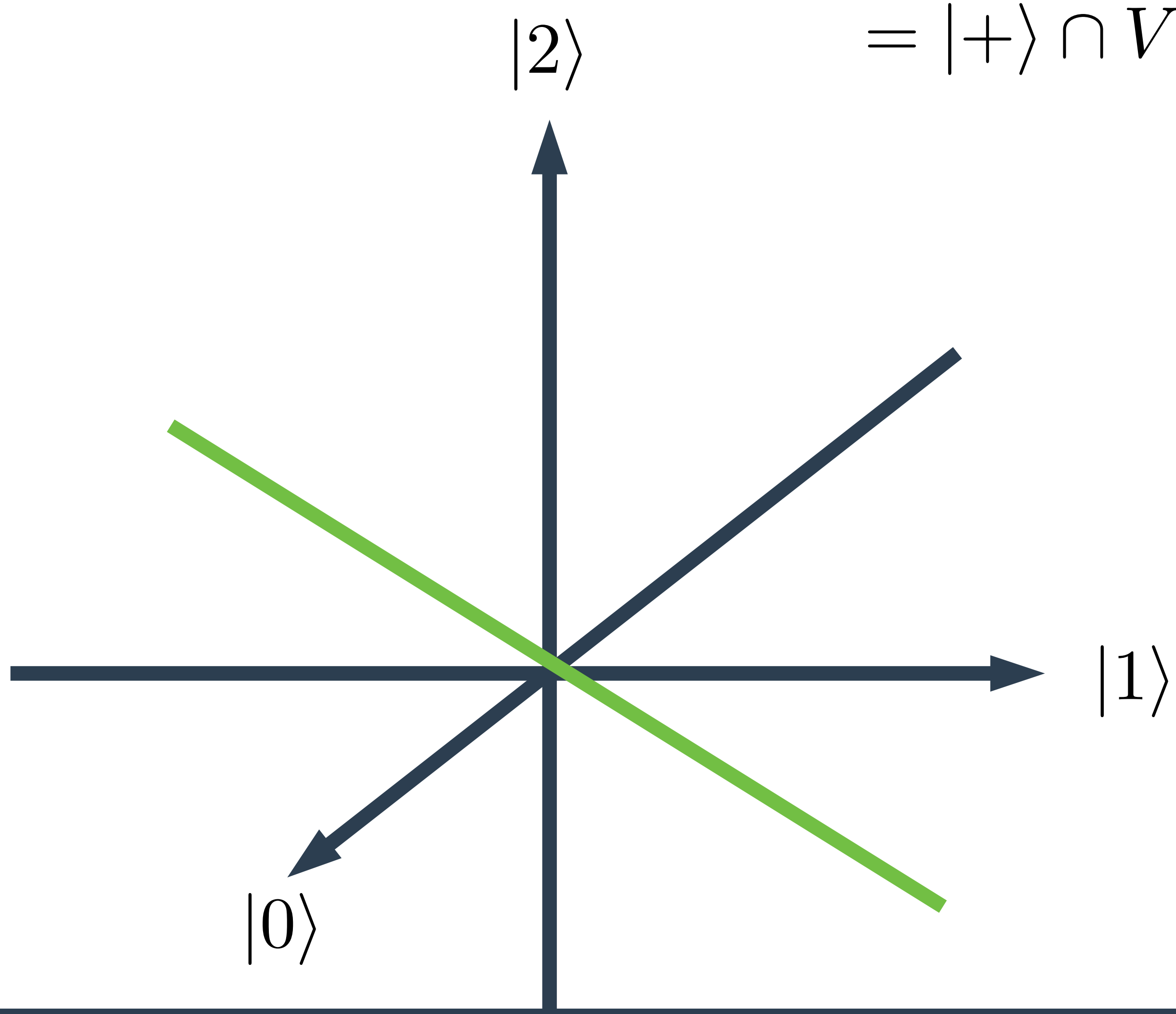
Non-distributivity of quantum logic



$$p = |+\rangle$$
$$q = \mathrm{span}(|0\rangle, |2\rangle)$$
$$r = |1\rangle$$

Non-distributivity of quantum logic

$$p \text{ and } (q \text{ or } r) = |+\rangle \cap \text{ span}(\text{span}(|0\rangle, |2\rangle), |1\rangle))$$
$$= |+\rangle \cap V = |+\rangle$$



$$p = |+\rangle$$
$$q = \text{span}(|0\rangle, |2\rangle)$$
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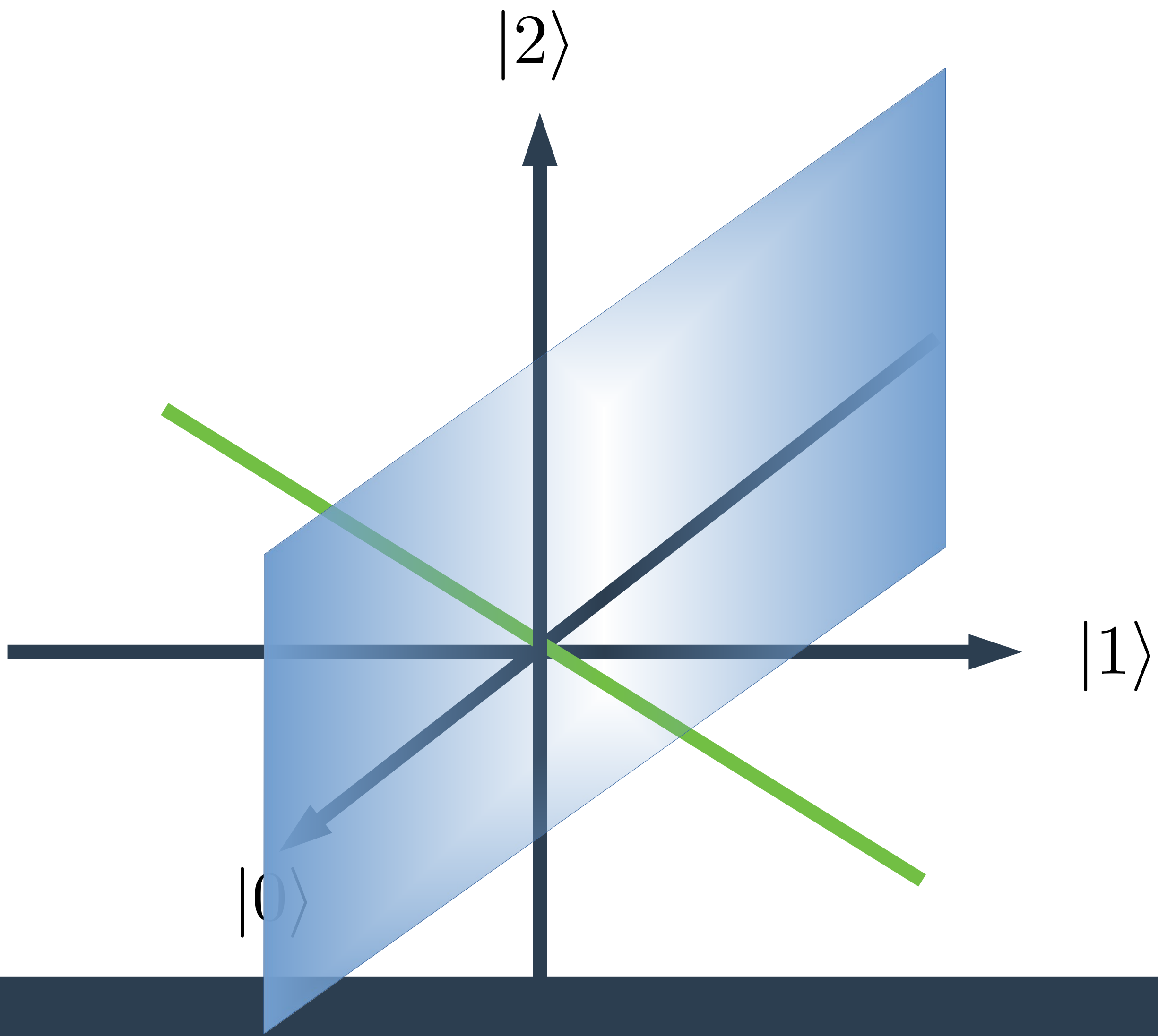
Non-distributivity of quantum logic



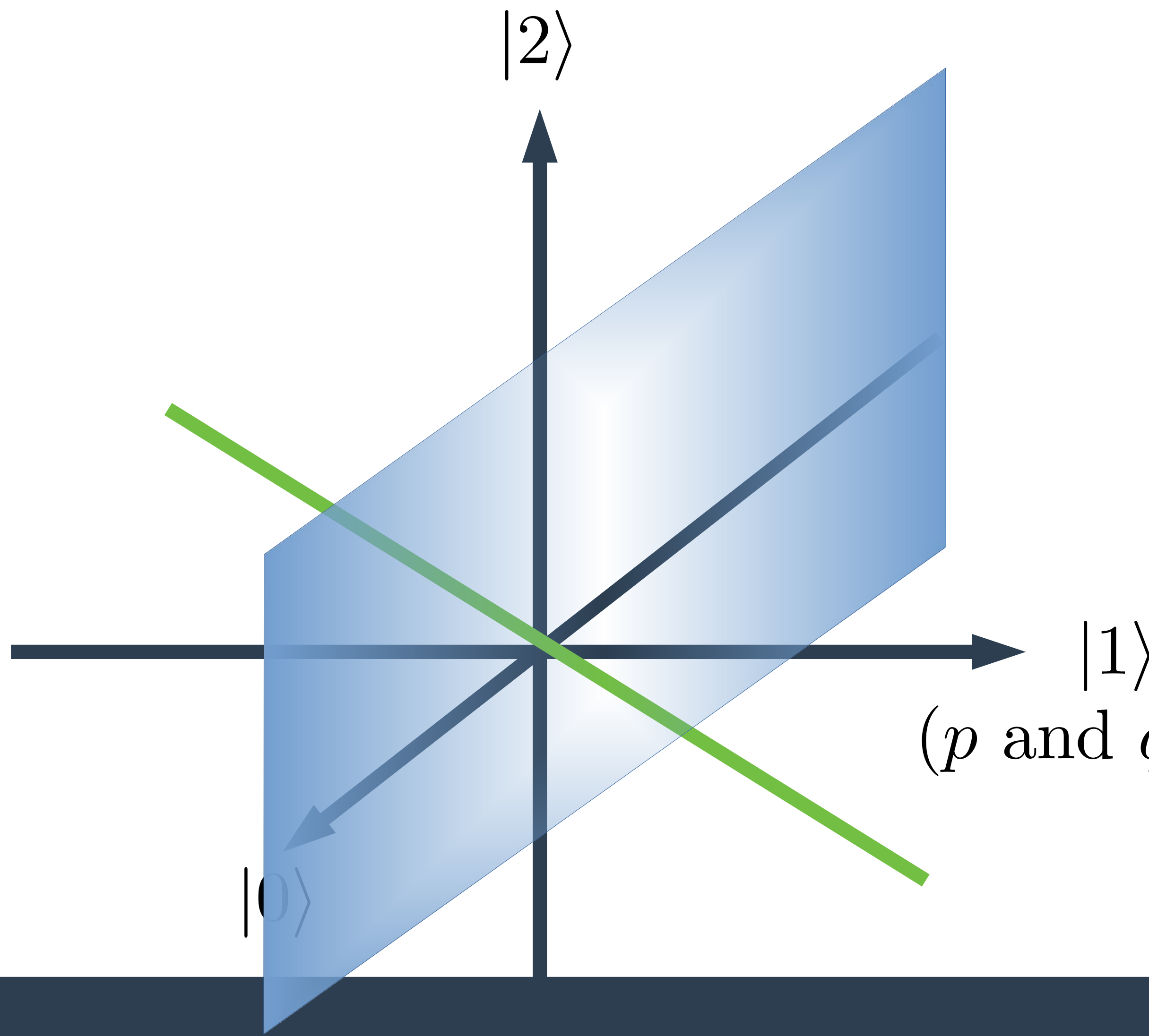
$$p = |+\rangle$$

$$q = \mathrm{span}(|0\rangle, |2\rangle)$$

$$r = |1\rangle$$

Non-distributivity of quantum logic

$(p \text{ and } q) \text{ or } (p \text{ and } r)$

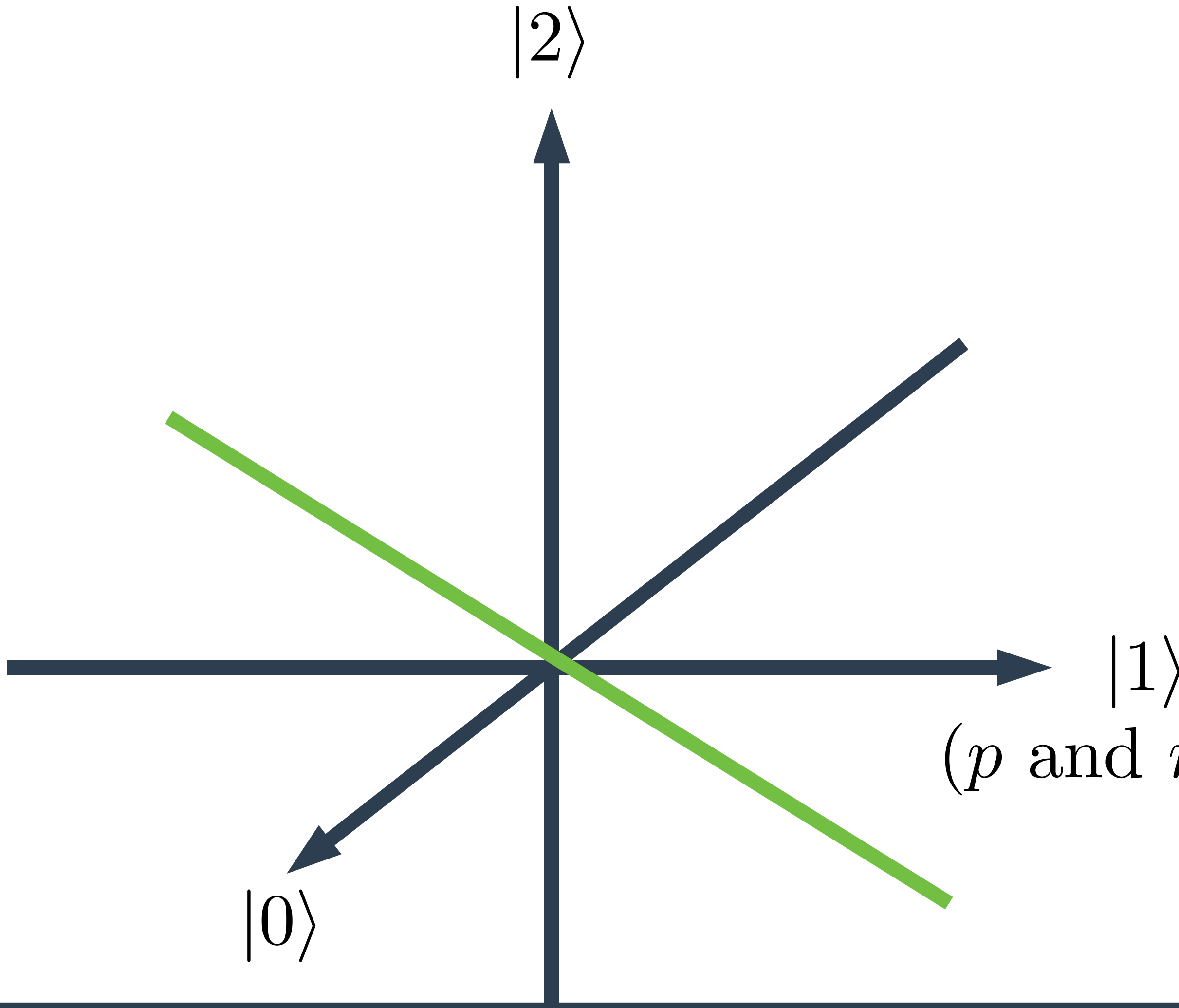


$$p = |+\rangle$$
$$q = \text{span}(|0\rangle, |2\rangle)$$
$$r = |1\rangle$$

$$(p \text{ and } q) = |+\rangle \cap \text{span}(|0\rangle, |2\rangle)$$
$$= \emptyset$$

Non-distributivity of quantum logic

$(p \text{ and } q) \text{ or } (p \text{ and } r)$



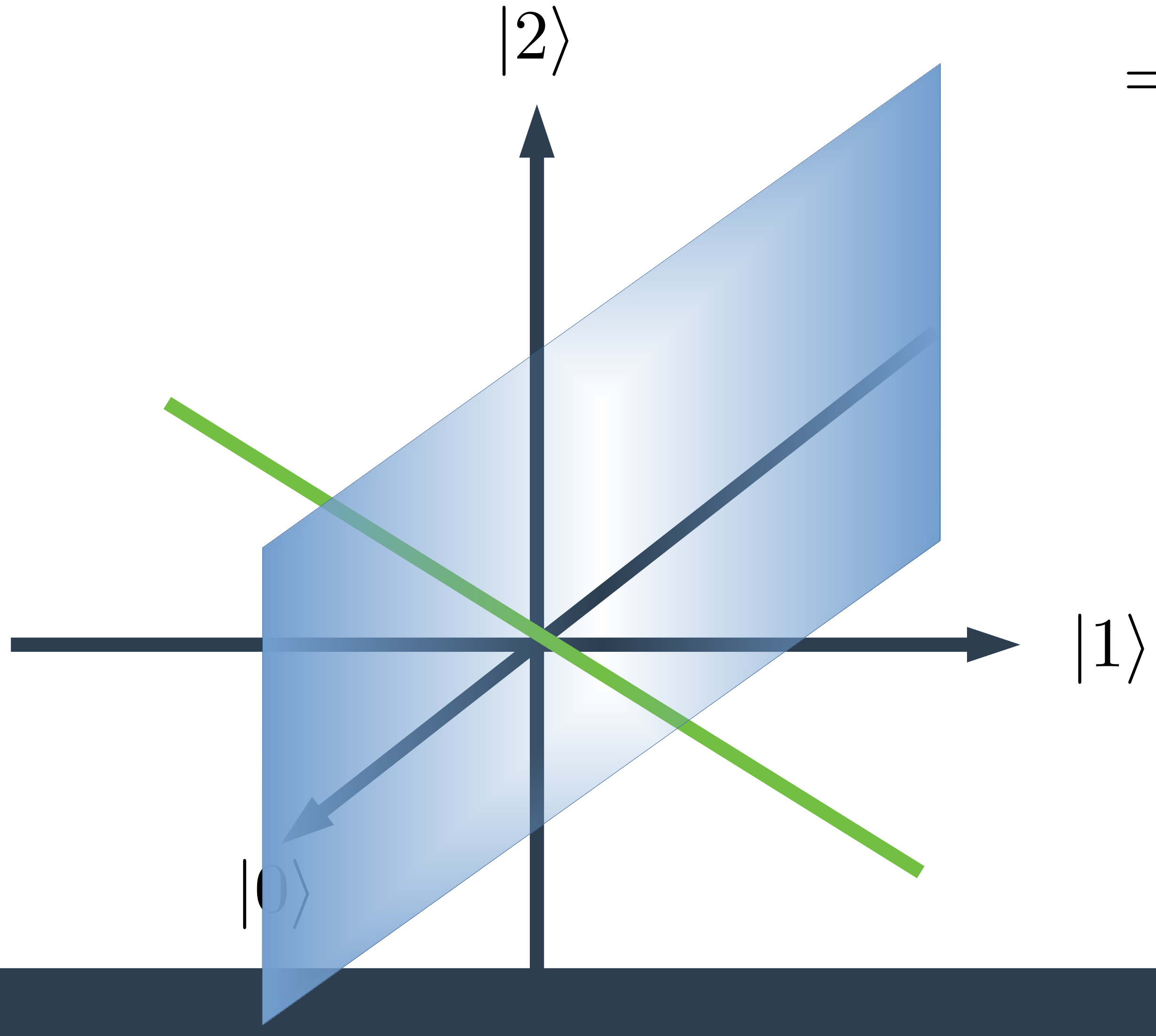
$$p = |+\rangle$$
$$q = \text{span}(|0\rangle, |2\rangle)$$
$$r = |1\rangle$$

$$(p \text{ and } r) = |+\rangle \cap |1\rangle$$
$$= \emptyset$$

Non-distributivity of quantum logic

$$(p \text{ and } q) \text{ or } (p \text{ and } r)$$

$$= \text{span}(\emptyset, \emptyset) = \emptyset$$



Non-distributivity of quantum logic

$$(p \text{ and } q) \text{ or } (p \text{ and } r) = \emptyset$$

$$p \text{ and } (q \text{ or } r) = |+\rangle$$

Operational quantum logic

- Propositions (subspaces) can be thought of as answers to yes/no questions (Mackey).
- These questions are the (projective) measurements we carry out.

Gleason's theorem

- If the answers correspond to subspaces, what are the possible probability assignments?
- Measure

$$\mu : P \to [0, 1]$$

- For mutually exclusive answers we expect probabilities to add: $P(A \text{ or } B) = P(A) + P(B)$

Gleason's theorem

- If the answers correspond to subspaces, what are the possible probability assignments?

$$\mu : P \to [0, 1]$$

$$\mu(\sum_i P_i) = \sum_i \mu(P_i)$$

- For P_i projectors onto orthogonal subspaces (mutually exclusive properties)

Gleason's theorem

The only measures on L(H) (for $d > 2$) which obey the lattice structure are of the form

$$\mu(A) = \mathrm{Tr}(\rho P_A)$$

Where ρ is a density operator

Axiomatisations of quantum theory

- Assume the structure of questions and answers
- The only probability rule compatible with the lattice structure of subspaces is the Born rule
- Recover the probability rule and states.

A sketch for a reconstruction

- Assume structure of questions
- Use Gleason's theorem to get states and probability rule
- Use theorem by Wigner to get the unitary group

Conclusion

- We adopted a certain perspective on quantum theory.
- Namely it is an Operational Quantum Logic
- Using this perspective we see that the axioms about the probability rule (and states) directly follow from the structure of projective measurements.

Conclusion

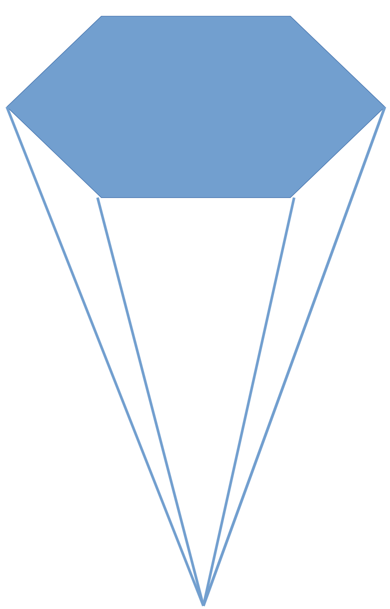
- In classical physics experimental propositions can be identified with sets
- In quantum physics experimental propositions can be identified with subspaces
- Both have a lattice structure

Where next?

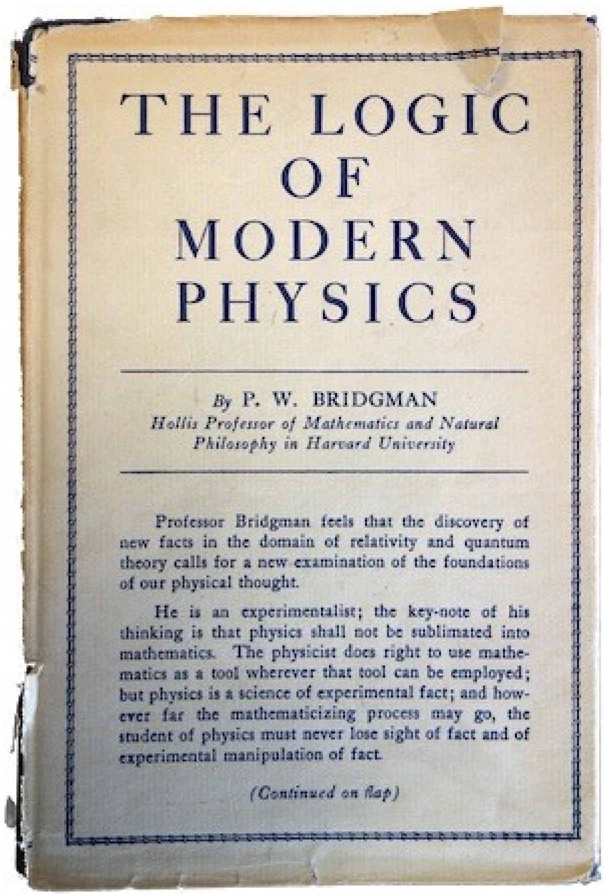
- We will adopt a different operational perspective
- That of General Probabilistic Theories
- Using this perspective we see that the axioms about the probability rule (and measurements) directly follow from the structure of pure states.

$$p(i|P) = \vec{e}_i \cdot \vec{s}_P$$

General probabilistic theories



Operationalism (1927)



https://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/significance_2/index.html

Operationalism

- "We mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations." Bridgman (1927)
- For instance concept of length should reduce to a set of operations one can carry out with a ruler.

Operationalism

- Special relativity
- Theories defined via a small number of operational principles.
- Operational reconstructions of quantum theory
- See Lluis Masanes lectures from last summer school

General probabilistic theories

Quantum Theory From Five Reasonable Axioms

Lucien Hardy*
Centre for Quantum Computation,
The Clarendon Laboratory,
Parks road, Oxford OX1 3PU, UK

February 1, 2008

The operational approach to physical theories

- In the operational approach a theory just allows us to make predictions about the outcomes of measurements.
- No claims are made about ontology or underlying physical reality.
- We have access to some physical devices which we can wire together, and the theory allows us to make predictions about what will happen.

The operational approach to physical theories

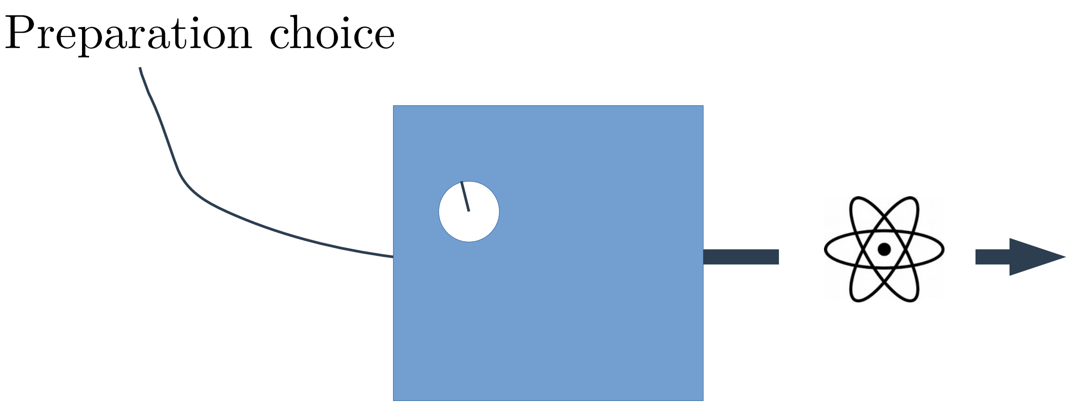
- Operational principles will lead to certain mathematical structure
- Mixing: convex structure
- Composition of devices: categorical structure

Primitives

- The primitives are devices: preparation, transformation and measurement devices.

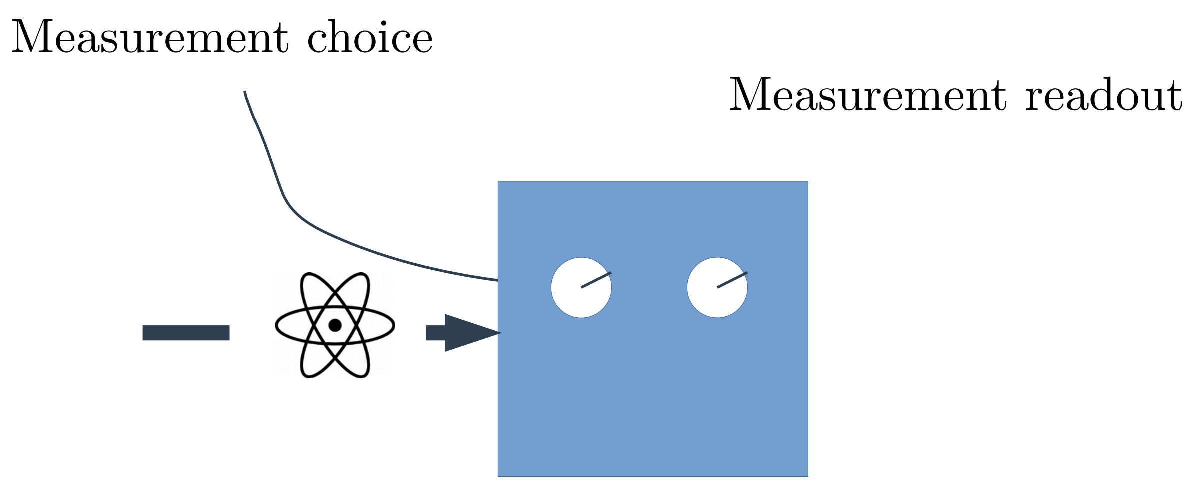
The operational approach

- The primitives are devices: preparation, transformation and measurement devices.



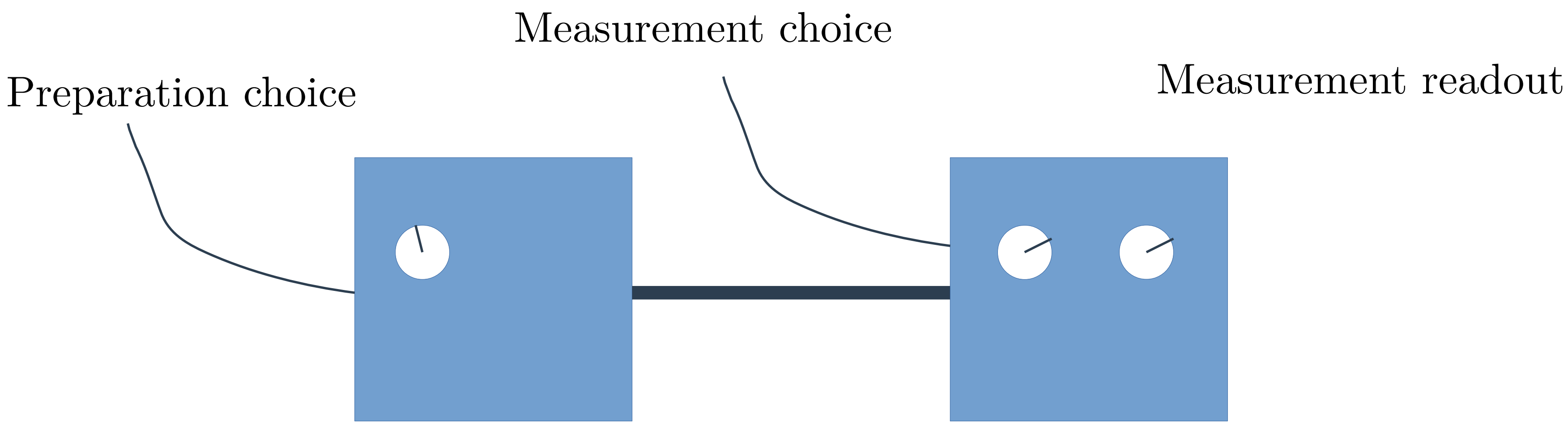
Measurement device

- The primitives are devices: preparation, transformation and measurement devices.

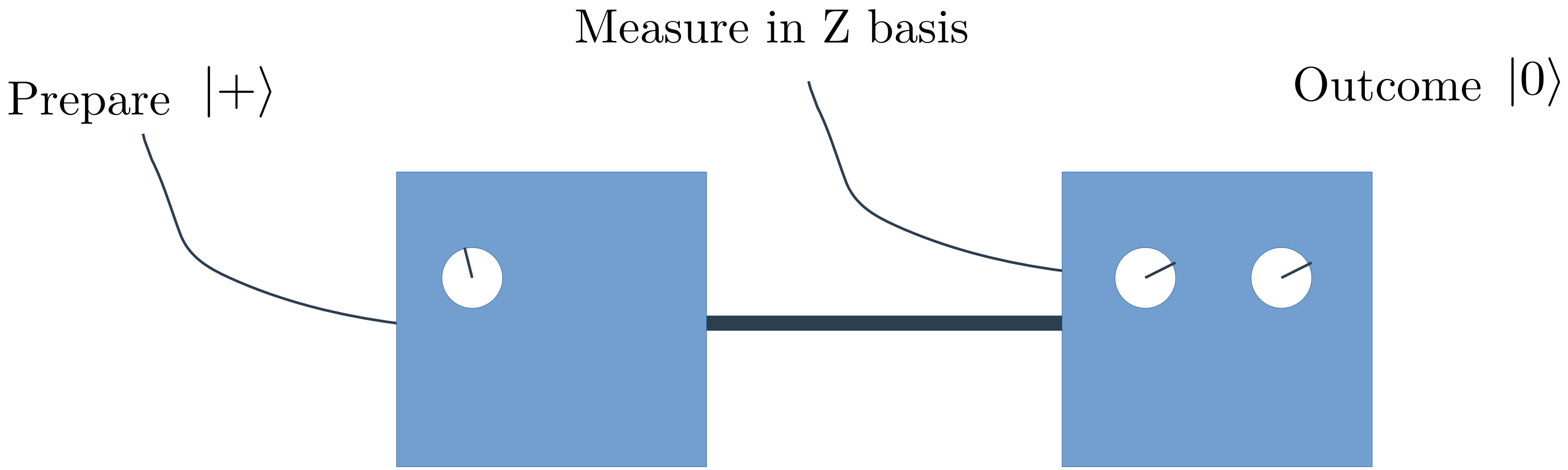


Prepare and measure

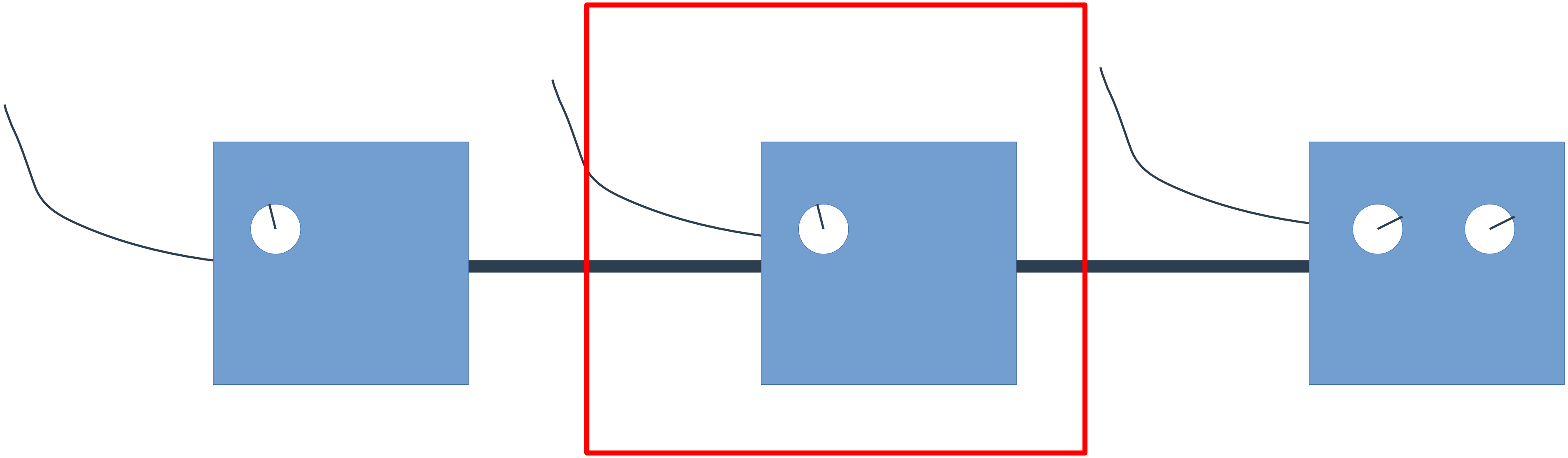
- The primitives are devices: preparation, transformation and measurement devices.



An example: qubit



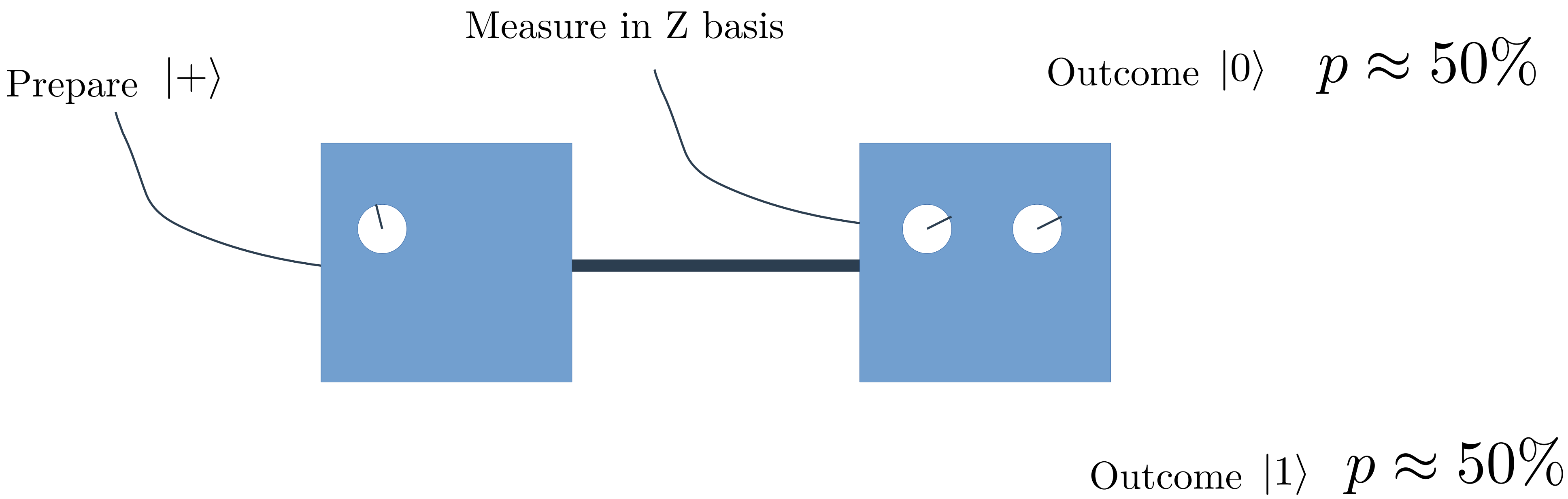
Transformation devices



The operational approach in quantum foundations

- For various different settings of the preparation device we collect the statistics for various measurements.
- We use these statistics to generate the state space and effect space.

An example: qubit



The operational approach in quantum foundations

- The classical world: laboratories, agents and classical probability theory is assumed.
- Fundamentally not about systems, but about devices, and the correlations between classical inputs and outputs of these devices.
- From the collected statistics we derive the state and effect space.

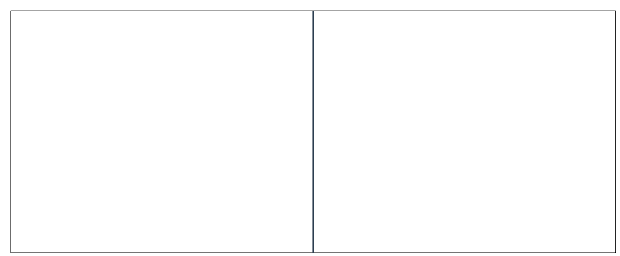
Classical probability theory (finite dimensional)

wikipedia

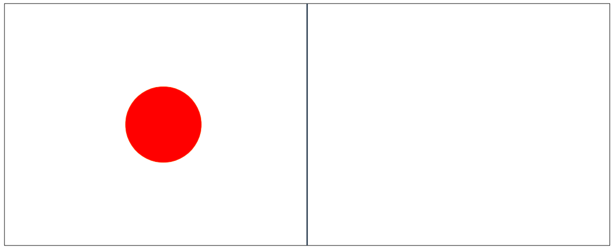
Getty images

Classical systems (finite dimensions)

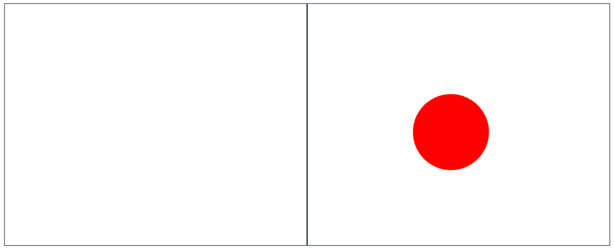
- Consider a ball which can be placed in one of two boxes:



Classical systems (finite dimensions)

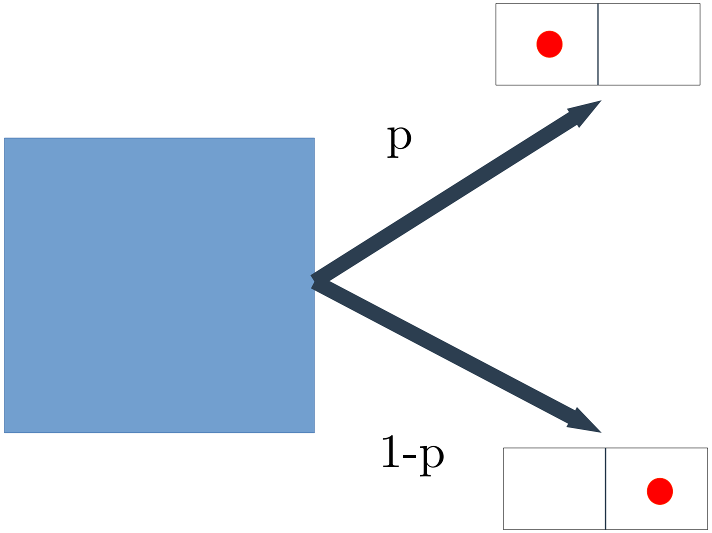


Classical systems (finite dimensions)



Preparations

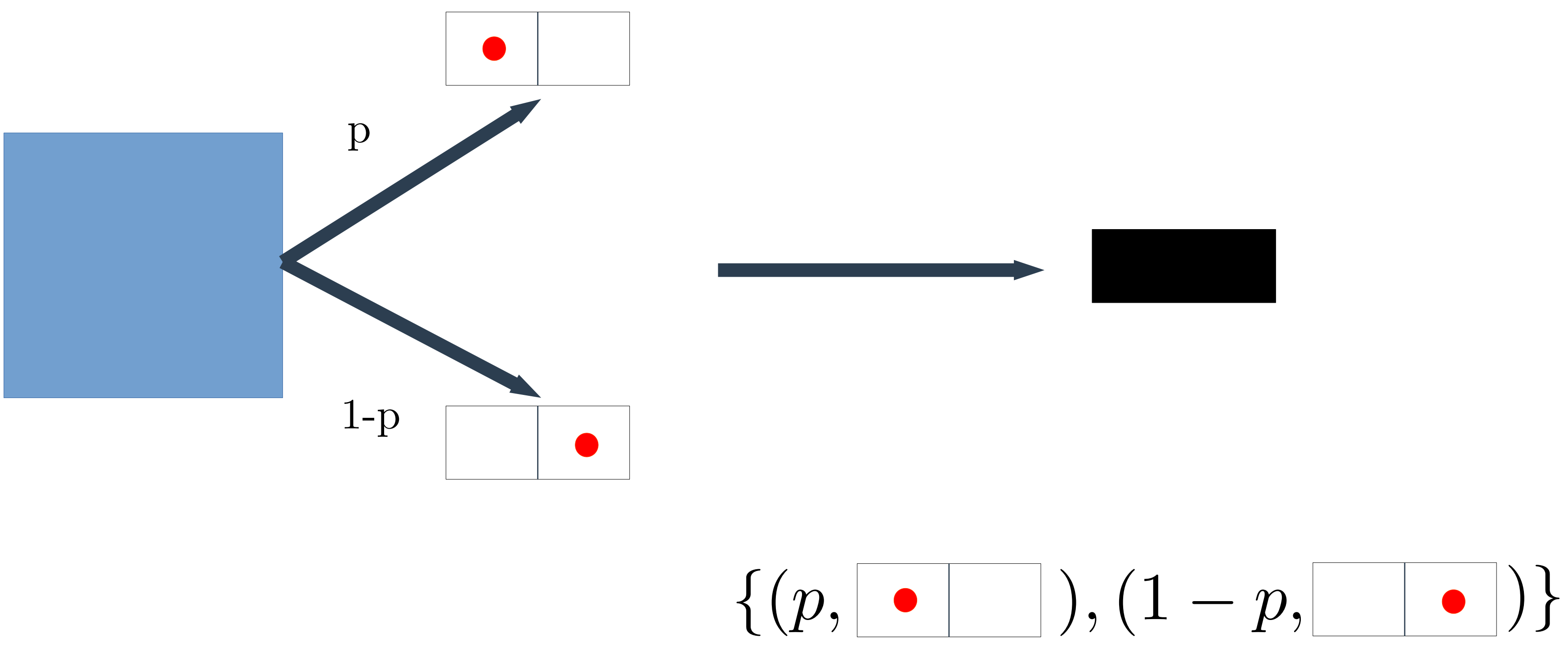
- A preparation consists of putting the ball in one of the boxes and closing them.
- I can choose to put the ball in box 1 with some probability p, and box 2 with probability 1-p



Preparations

- A preparation consists of putting the ball in one of the boxes and closing them.
- I can choose to put the ball in box 1 with some probability p, and box 2 with probability 1-p
- This is called preparing an ensemble, and is a central assumption of the GPT framework

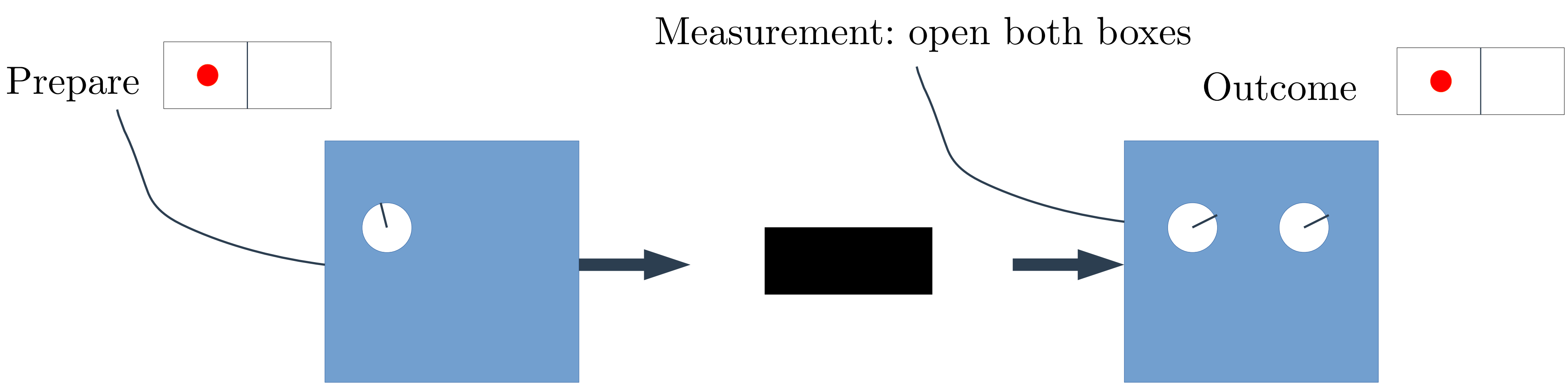
Preparing an ensemble



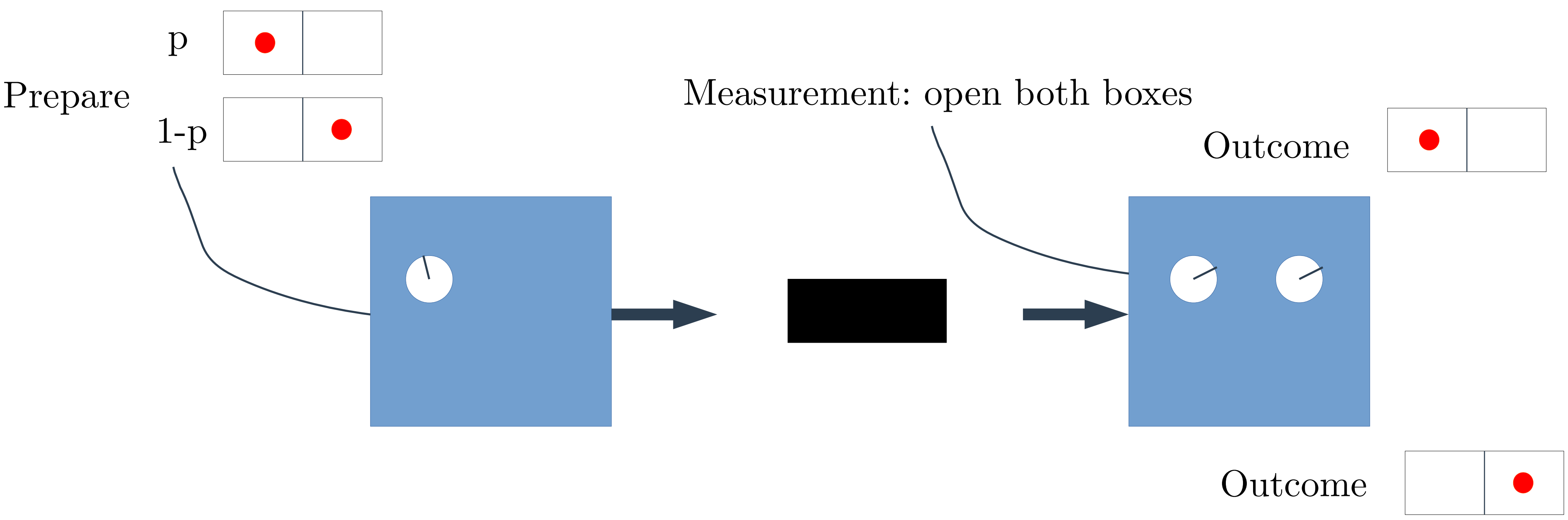
Measurements

- I give the closed box to someone else
- They can carry out a measurement: open the box to see which partition the ball is in
- For a given preparation we repeat this many times
- The measurement has two outcomes: "The ball is in box 0" and "The ball is in box 1".

An example



An example



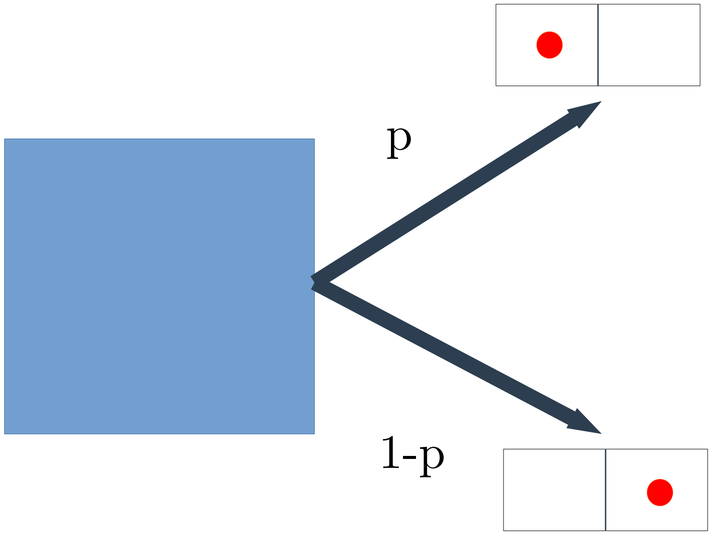
States

- For a given preparation outcome 0 will occur with probability p, and outcome 1 with probability 1-p

$$\vec{s}_P = \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

State just allows us to make predictions.

States

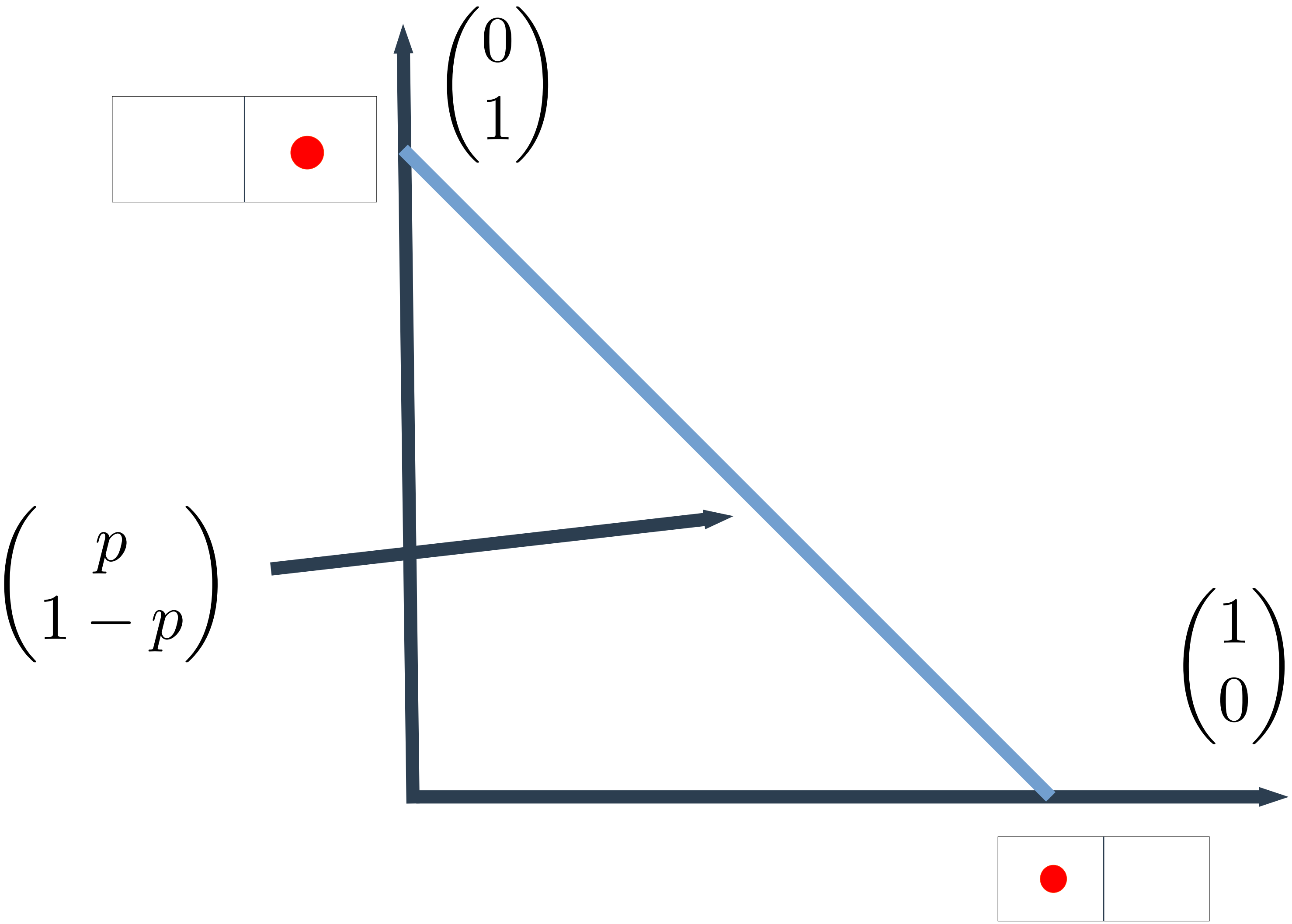


$$\vec{s}_P = \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

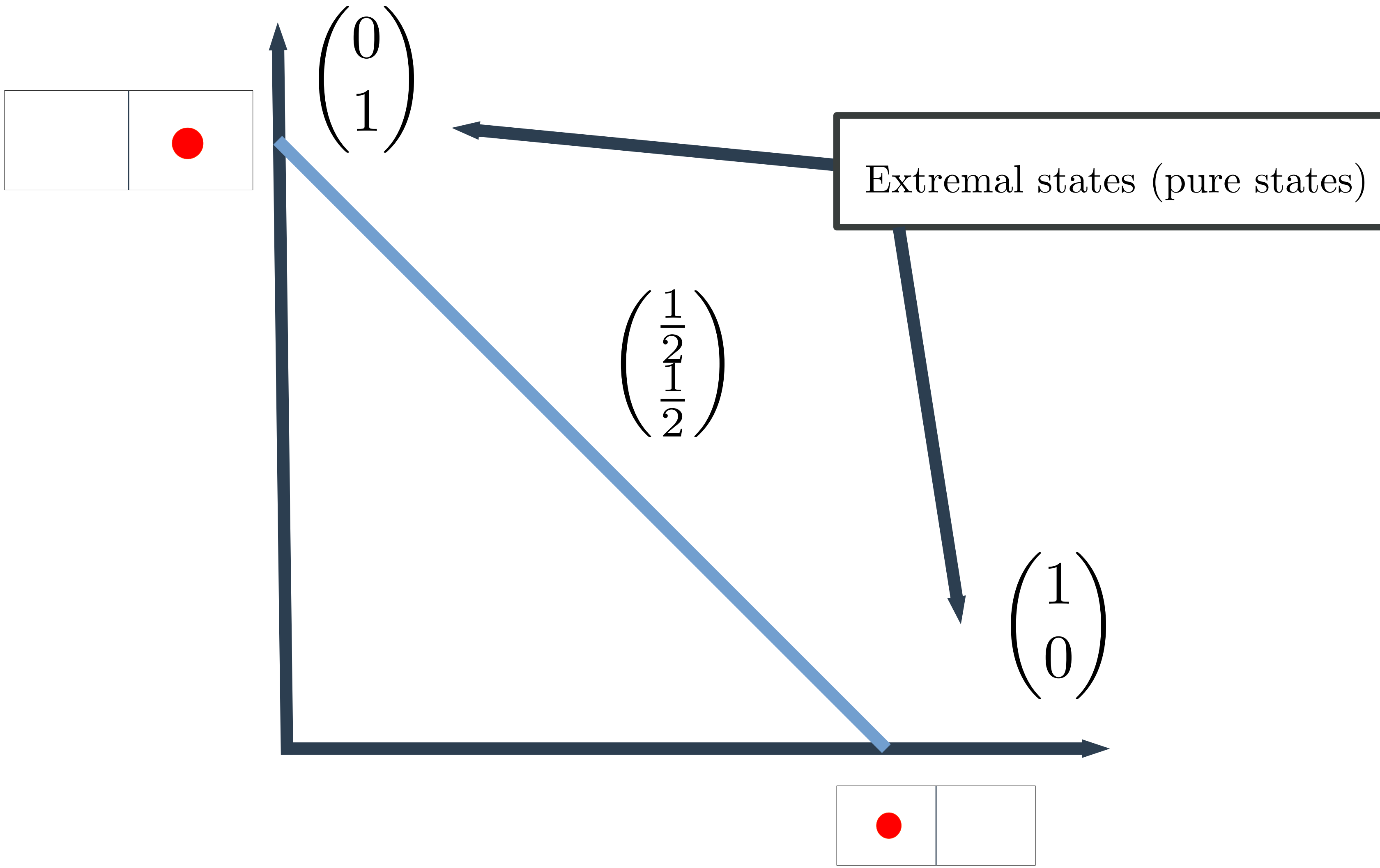
States

- A state is a probability distribution over outcomes
- Just like in operational quantum logic

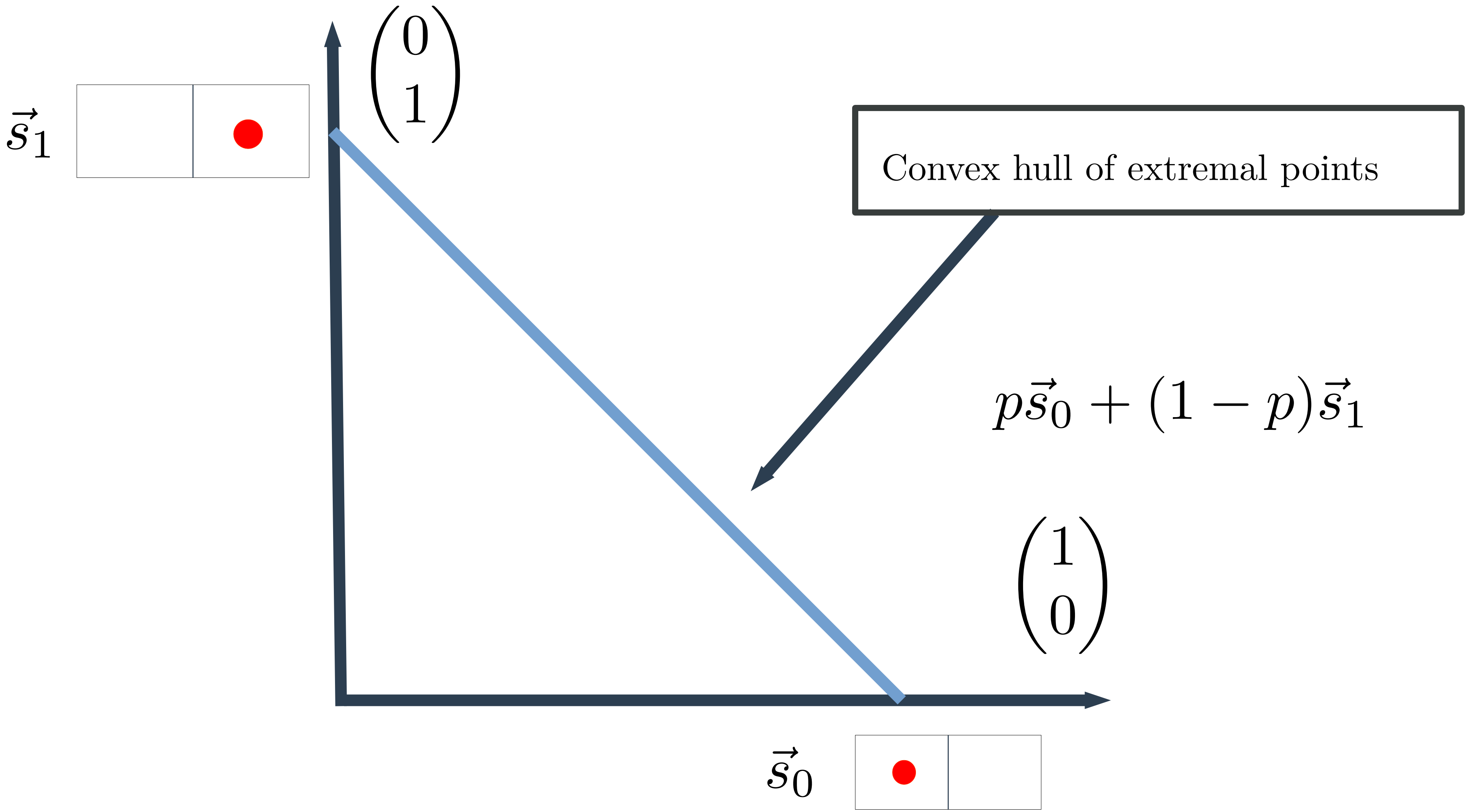
State space


$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} p \\ 1-p \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

State space


$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Extremal states (pure states)
$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

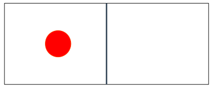
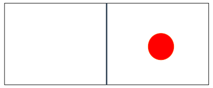
State space



$\vec{s}_1$
$\begin{pmatrix}0\\1\end{pmatrix}$
Convex hull of extremal points
$p\vec{s}_0 + (1-p)\vec{s}_1$
$\begin{pmatrix}1\\0\end{pmatrix}$
$\vec{s}_0$

Convex combination

$$p\vec{s}_0 + (1-p)\vec{s}_1$$

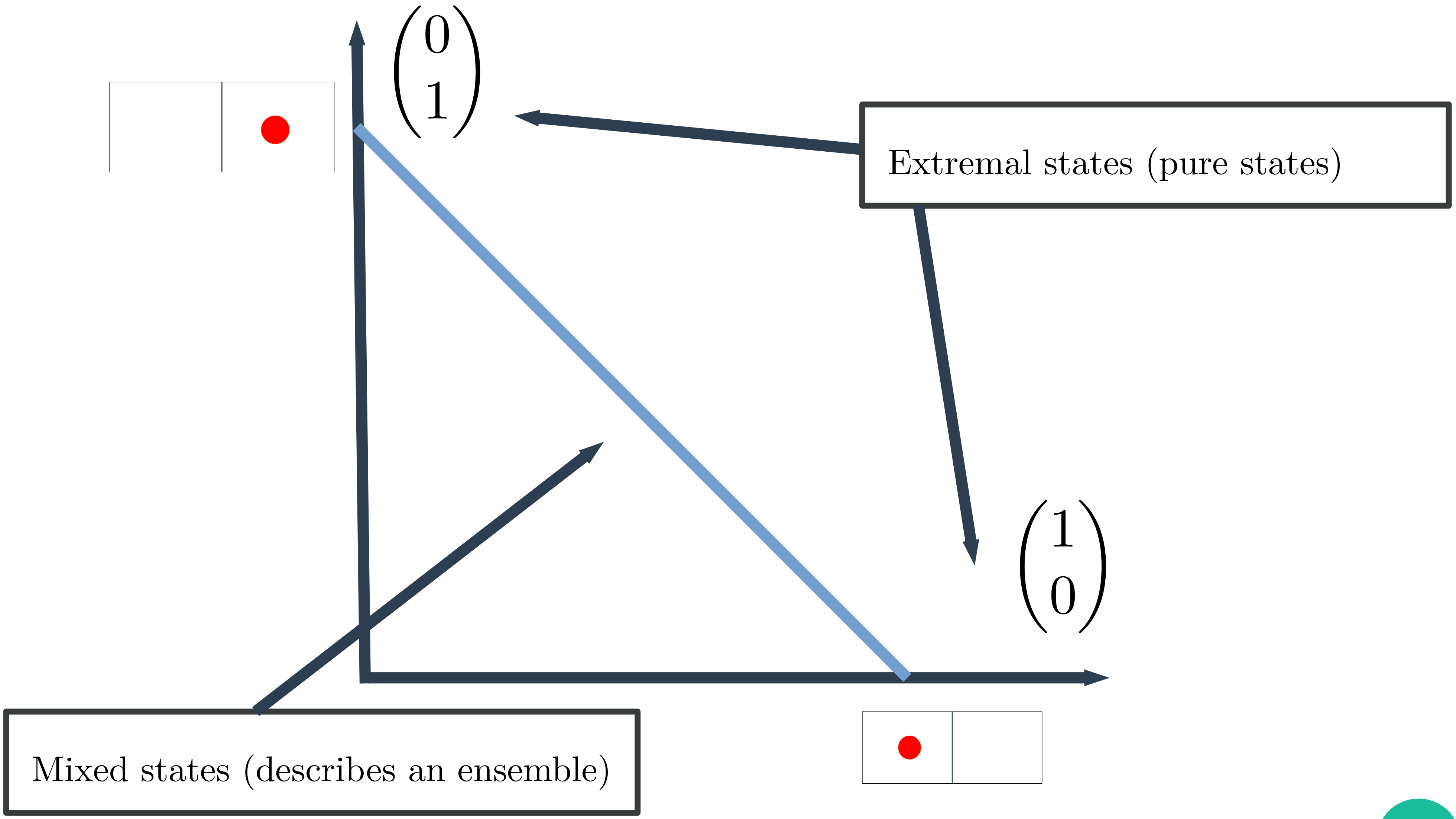
$\{(p,$  $), (1-p,$  $)\}$

$$\sum_i p_i \vec{s}_i$$

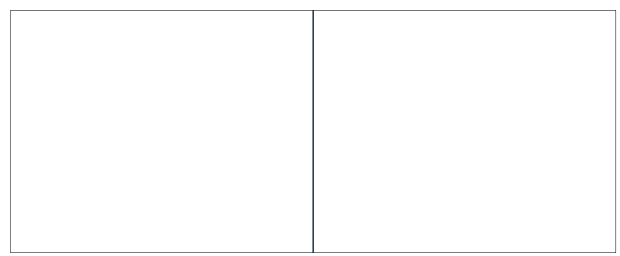
$$p_i \geq 0$$

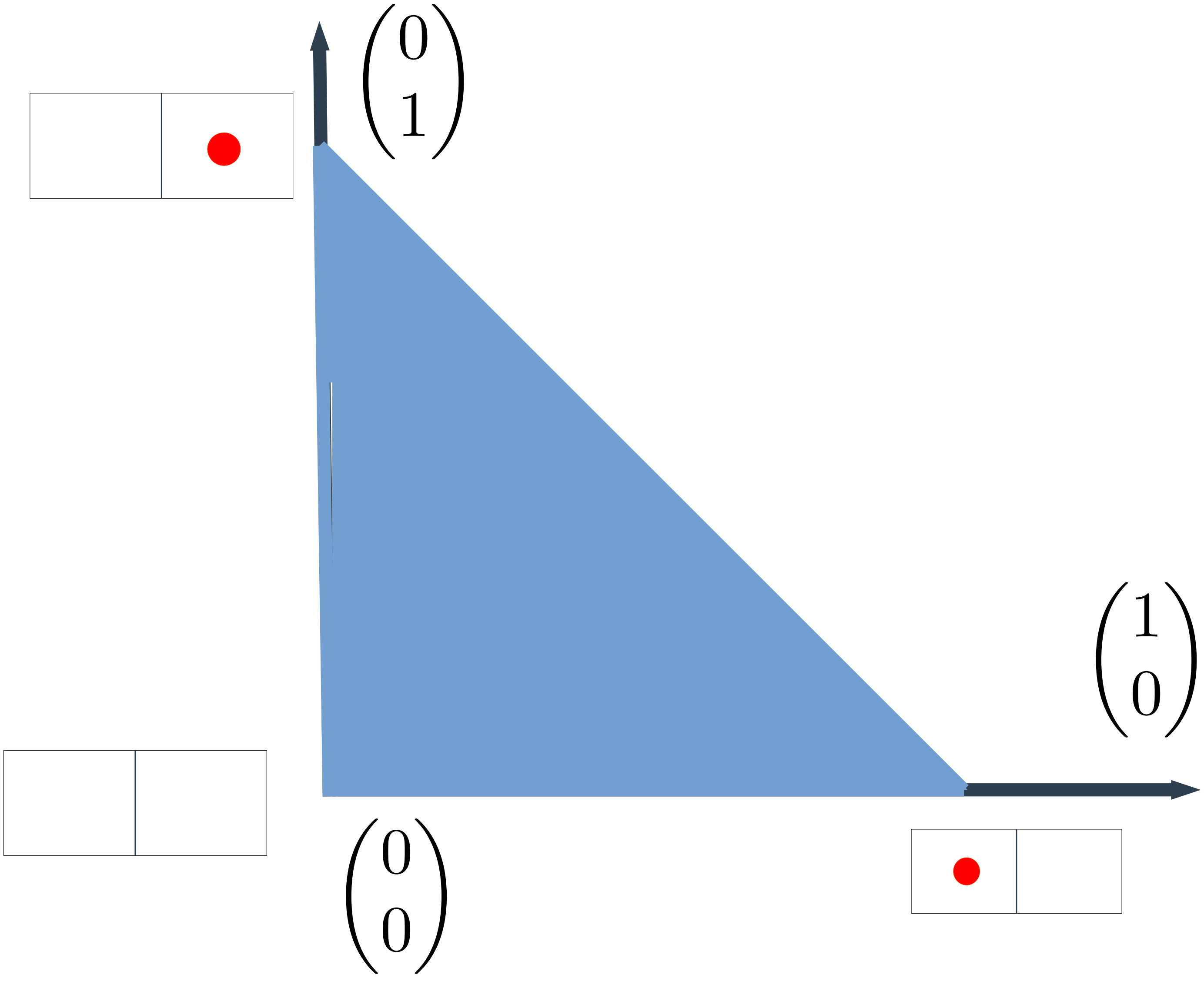
$$\sum_i p_i = 1$$

State space


$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Extremal states (pure states)
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Mixed states (describes an ensemble)

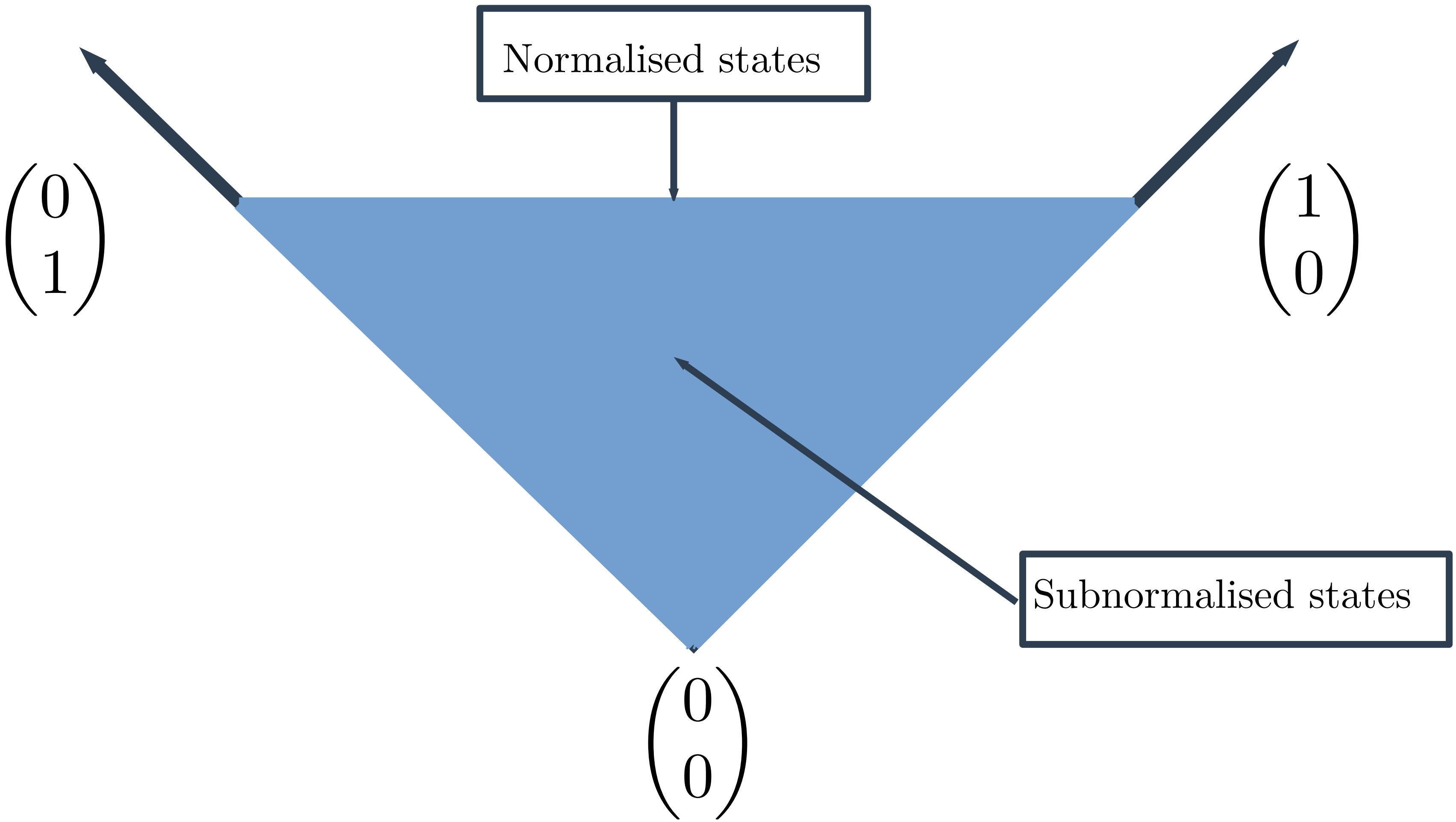
Subnormalised states

- The experimenter can also choose to just send an empty box
- 
-
- This will give probability 0 for all measurement outcomes, and hence is represented by
- $$\vec{s}_E = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}0\\0\end{pmatrix}$

Cone of states



Effects

- How do I obtain the probability p(0|P) of obtaining outcome 0 from the state $\vec{s}_P$?

$$p(0|P) = \vec{e}_0 \cdot \vec{s}_P = (1,0) \cdot \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix}$$

$$p(1|P) = \vec{e}_1 \cdot \vec{s}_P = (0,1) \cdot \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix}$$

Effects

$$p(0|P) = \vec{e}_0 \cdot \vec{s}_P = (1,0) \cdot \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix}$$

$$p(1|P) = \vec{e}_1 \cdot \vec{s}_P = (0,1) \cdot \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix}$$

What is probability of outcome 0 or outcome 1 occurring?

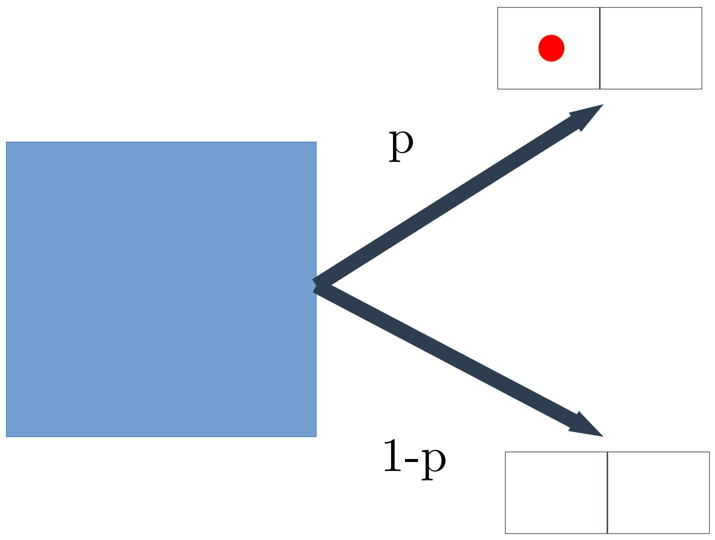
$$p(0 \text{ or } 1) = p(0|P) + p(1|P) = (1,1) \cdot \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix} = \vec{u} \cdot \vec{s}_P$$

What is probability of neither outcome occurring?

$$\vec{0} \cdot \vec{s}_P = (0,0) \cdot \begin{pmatrix} p(0|P) \\ p(1|P) \end{pmatrix}$$

Subnormalised states

- The experimenter can also prepare mixtures of the empty box and preparations with a ball in one of the partition

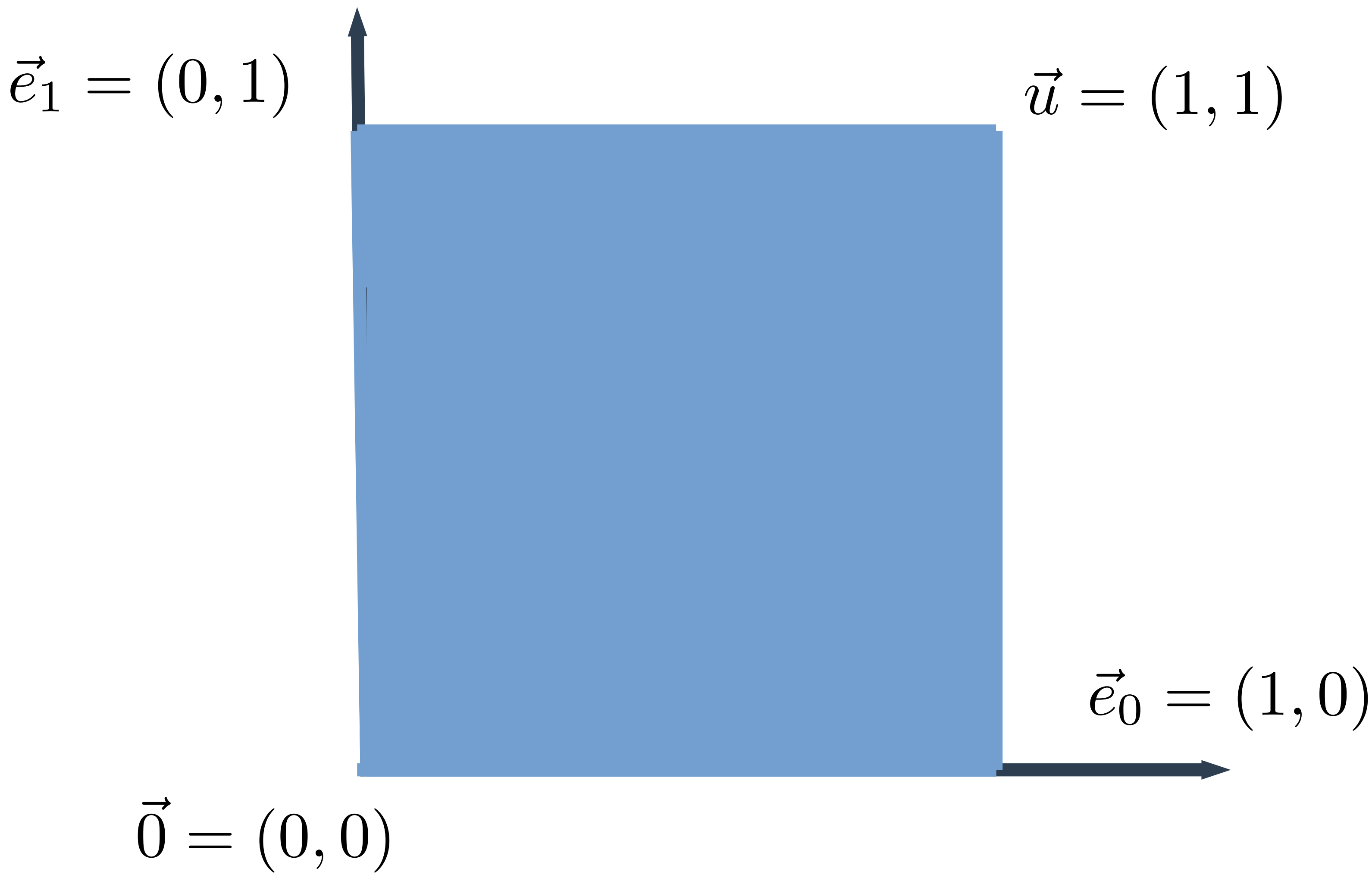


Effects

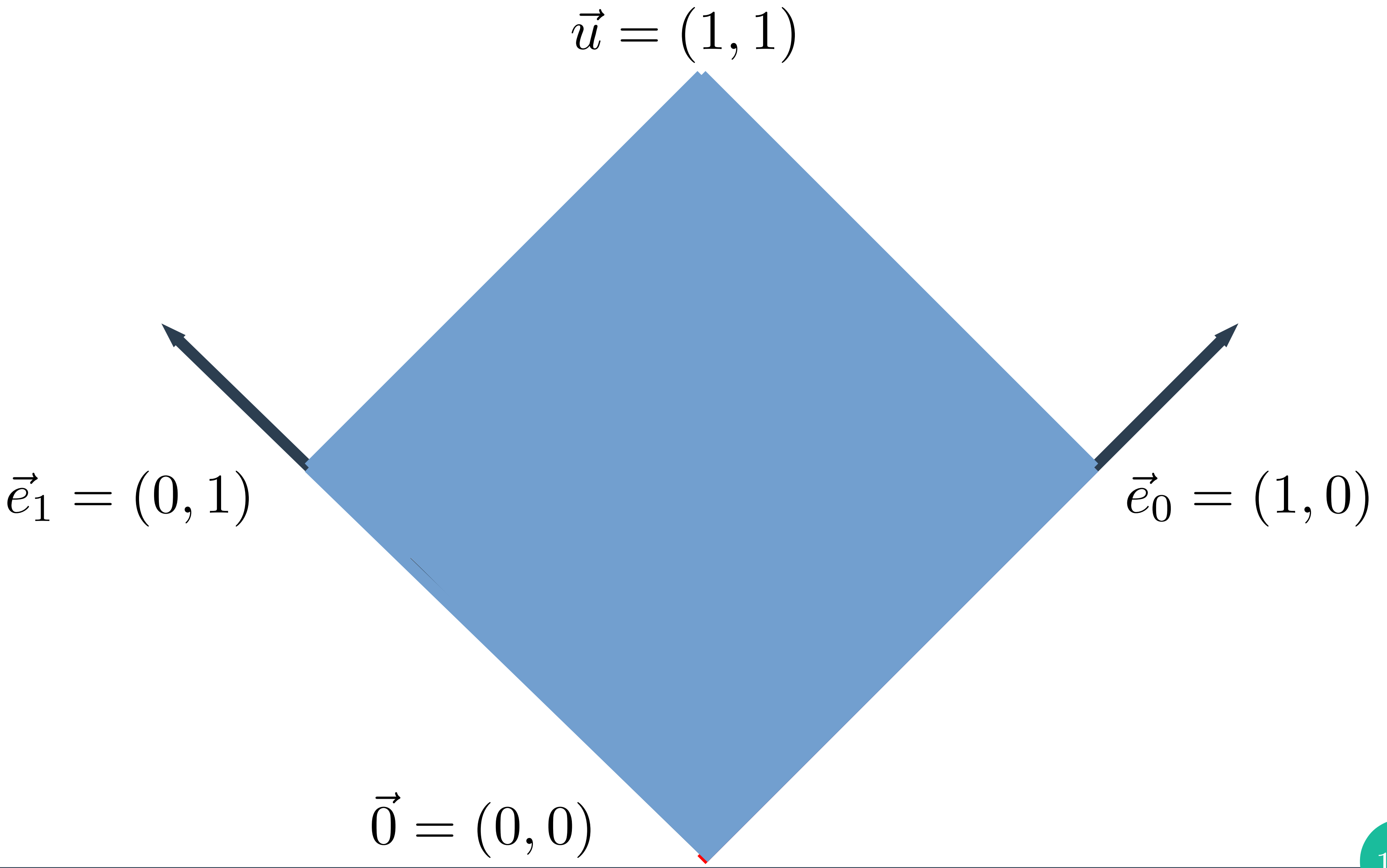
$$p(i|P) = \vec{e}_i \cdot \vec{s}_P$$

$\vec{e}_i$ is an effect. It is a linear functional of states (dual vector).

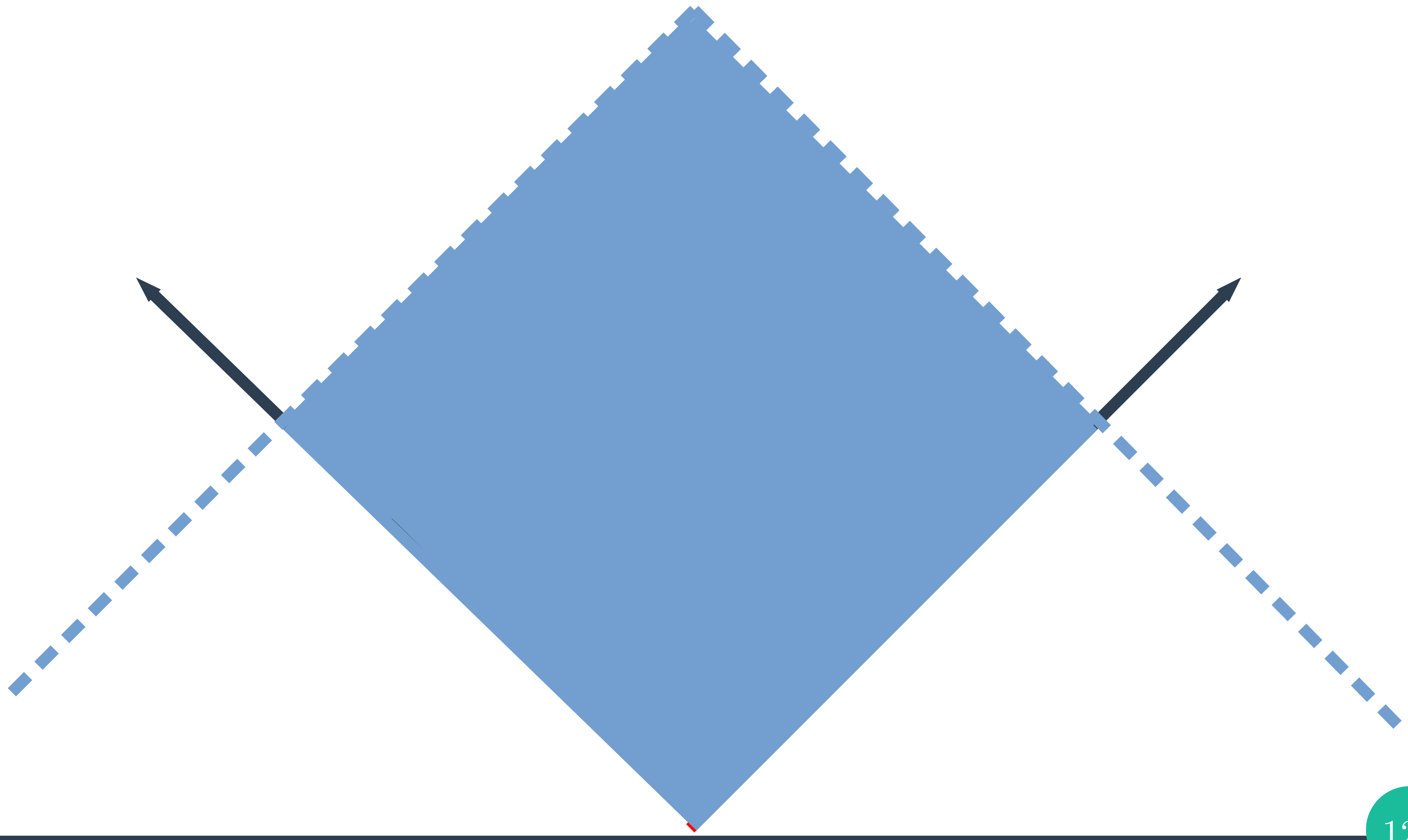
$$e : V \to \mathbb{R}$$


$\vec{e}_1 = (0, 1)$
$\vec{u} = (1, 1)$
$\vec{e}_0 = (1, 0)$
$\vec{0} = (0, 0)$

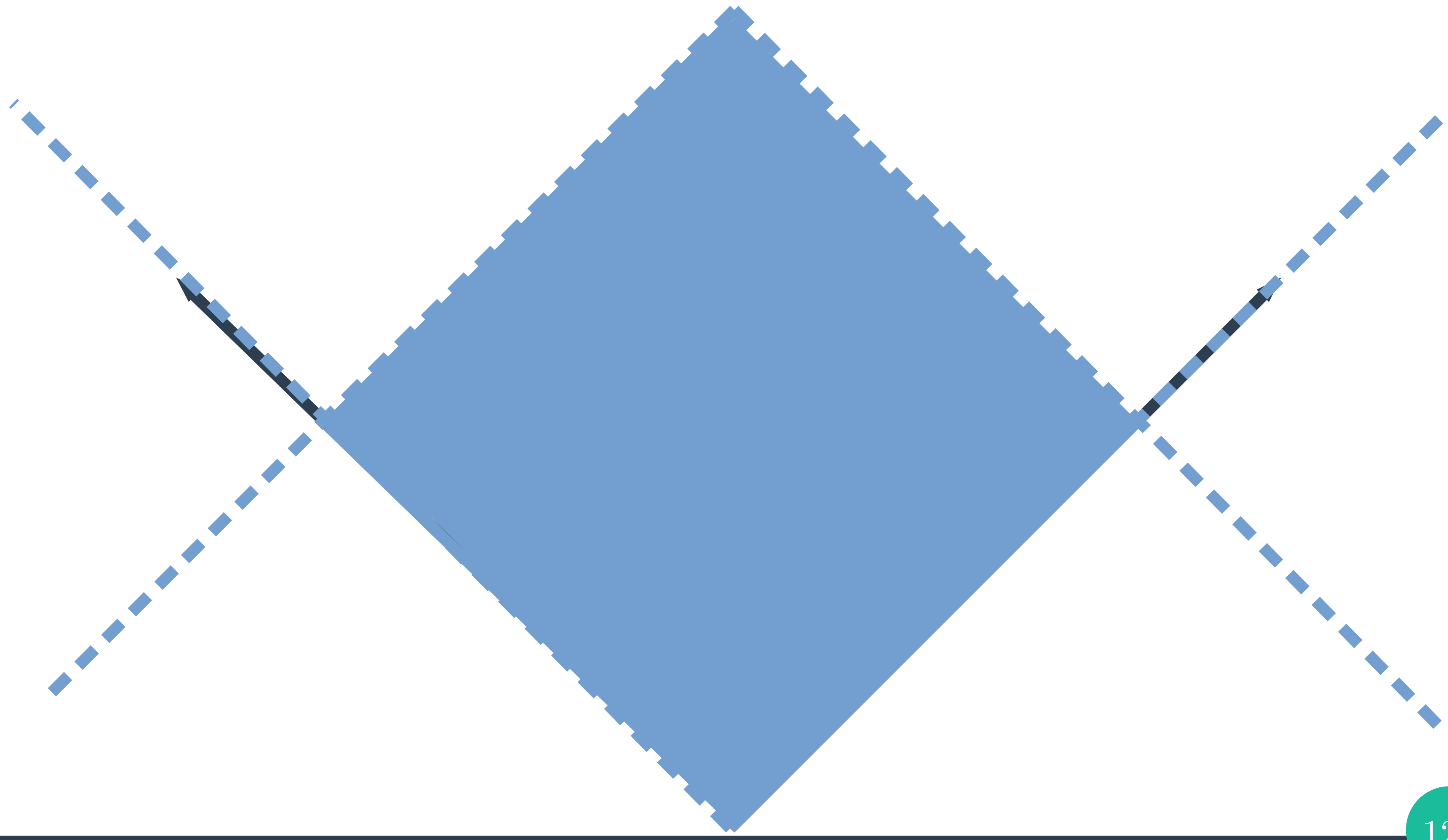
Effect space



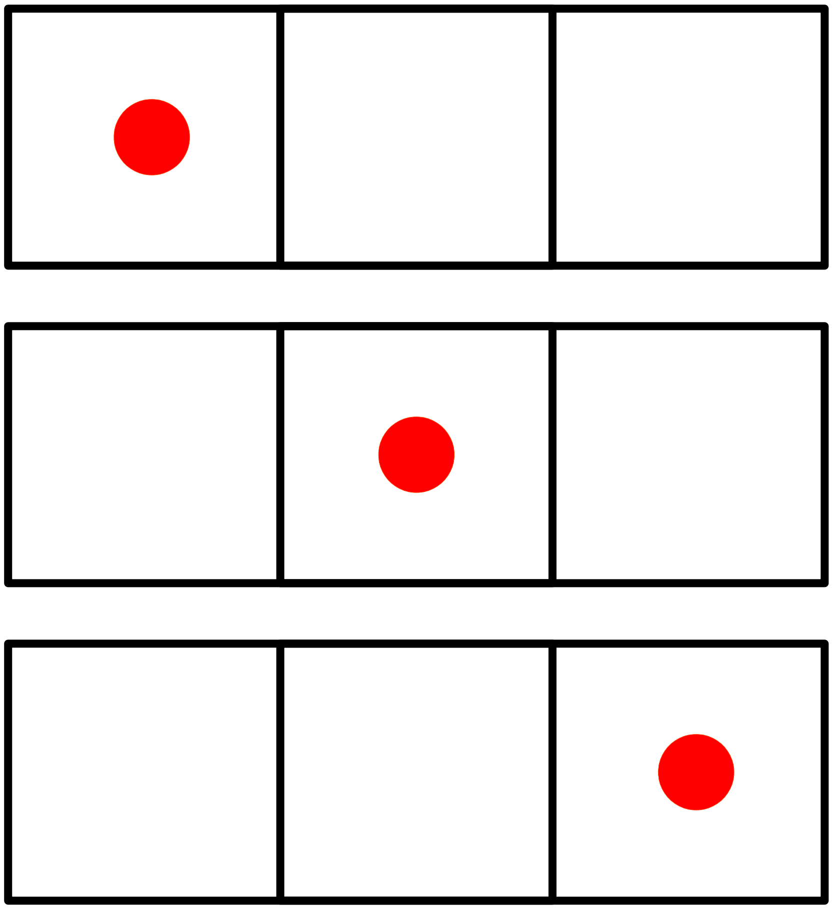
Effect space



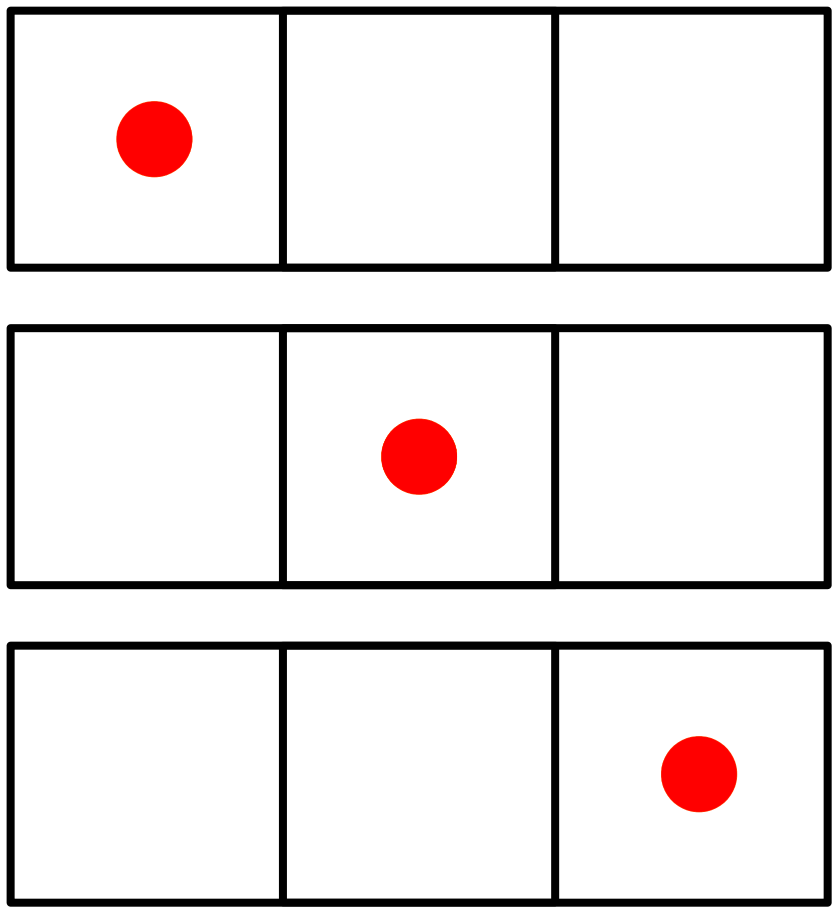
Effect space



The trit



The trit

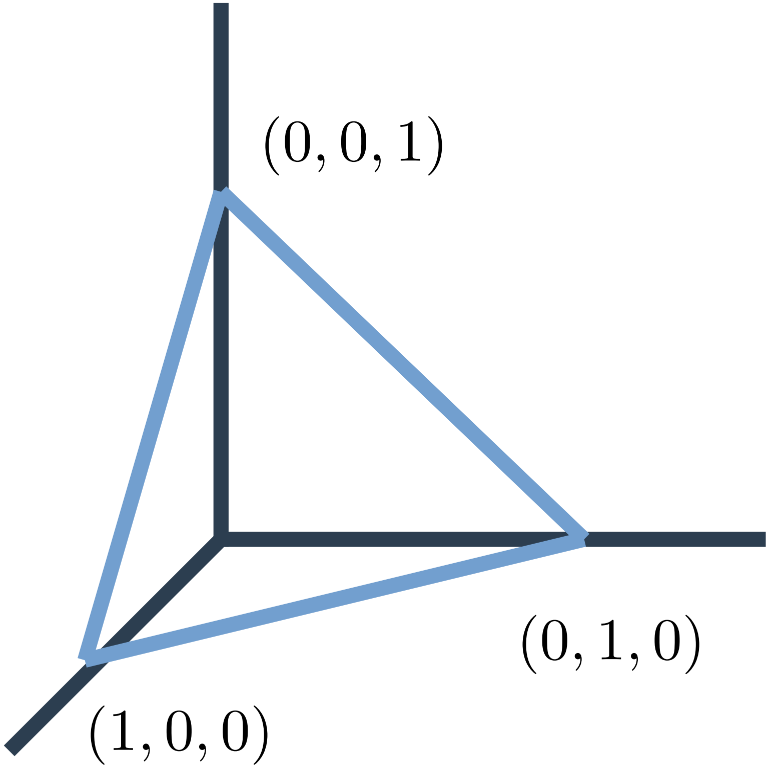


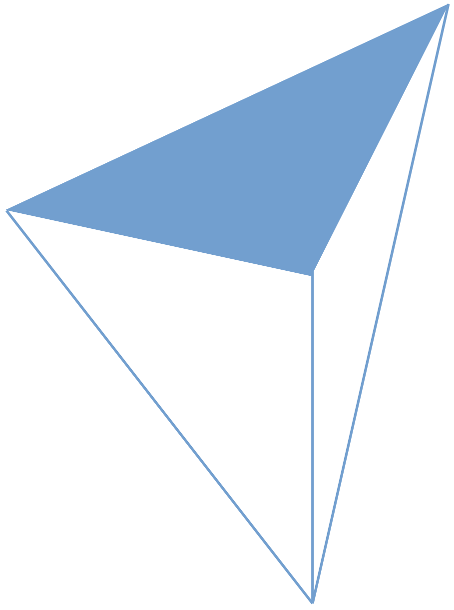
$(1, 0, 0)$

$(0, 1, 0)$

$(0, 0, 1)$

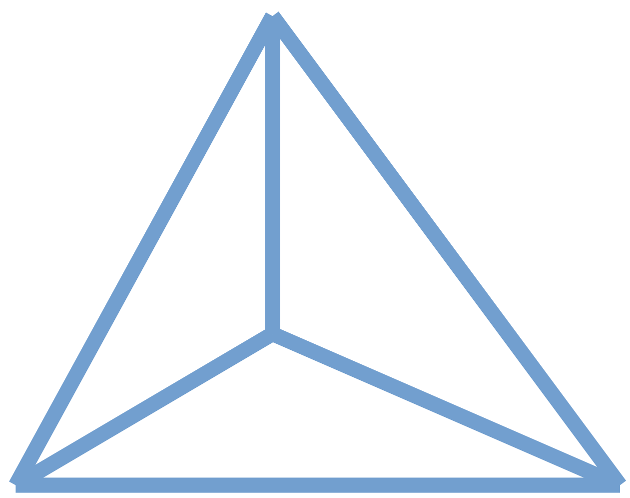
Trit state space

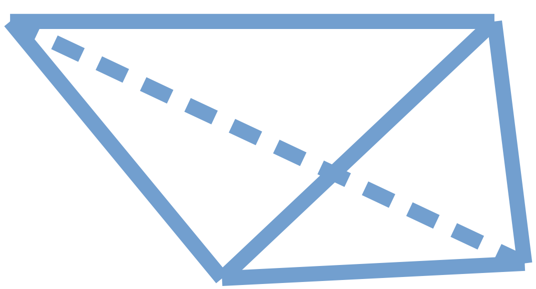




Classical four level systems

- Tetrahedron, embedded in $\mathbb{R}^4$, just drawing the normalised states





Classical systems

- A classical system with d pure states is a simplex with d extremal points.
- Line segment, triangle, tetrahedron...

Deriving the quantum state space

Qubit: deriving the state space

- Pure states $\psi \in \mathrm{P}\mathbb{C}^2$
- Two outcome measurements

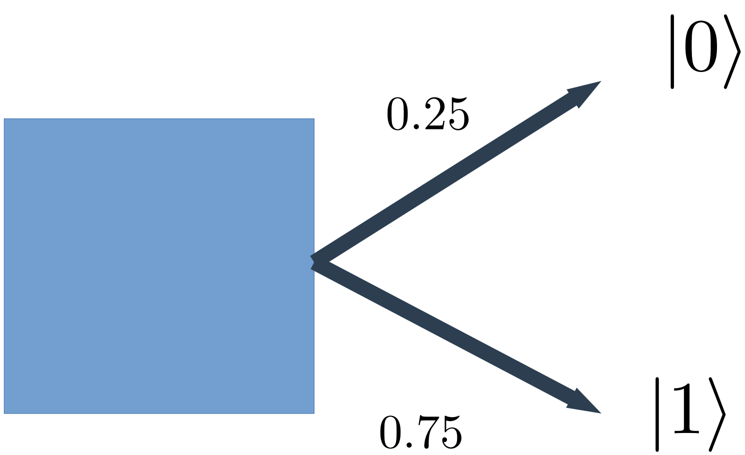
$$p(a|\psi) = |\langle a|\psi\rangle|^2$$

- Can prepare ensembles $P = \{p_i, \psi_i\}_i$

$$p(a|P) = \sum_i p_i p(a|\psi_i) = \sum_i p_i |\langle a|\psi_i\rangle|^2$$

Qubit: deriving the state space

$$P = \{(0.25, |0\rangle), (0.75, |1\rangle)\}$$



$$p(0|P) = 0.25 \times |\langle 0|0\rangle|^2 + 0.75 \times |\langle 0|1\rangle|^2 = 0.25$$

Qubit

Could write a state as a vector of probability distributions over outcomes

$$\vec{s}_\psi = \begin{pmatrix} p(a_1|\psi) \\ p(a_2|\psi) \\ p(a_3|\psi) \\ \vdots \end{pmatrix}$$

Qubit

- Could write a state as a vector of probability distributions over outcomes.
- But would be infinitely long.
- And contains redundancies.

Qubit

- Any $p(a|\psi) = |\langle a|\psi\rangle|^2$ can be written as a linear combination of 4 functions of the form $|\langle a_i|\psi\rangle|^2$

$$p(a|\psi) = \sum_{i=1}^{4} c_i p(a_i|\psi)$$

- For instance we can have the +X,+Y,+Z and -Z outcome probabilities as the basis functions.
- Could be a SIC-POVM

The probabilistic representation

$$p(a|\psi) = \sum_{i=1}^{4} c_i p(a_i|\psi) = \vec{e}_a \cdot \vec{s}_\psi$$

$$\vec{e}_a = (c_1, c_2, c_3, c_4)$$

$$\vec{s}_\psi = \begin{pmatrix} p(a_1|\psi) \\ p(a_2|\psi) \\ p(a_3|\psi) \\ p(a_4|\psi) \end{pmatrix}$$

The probabilistic representation

$$\vec{s}_\psi = \begin{pmatrix} p(+X|\psi) \\ p(+Y|\psi) \\ p(+Z|\psi) \\ p(-Z|\psi) \end{pmatrix}$$

$$p(a|\psi) = \vec{e}_a \cdot \vec{s}_\psi$$

Any outcome probability can be computed as a linear function of the 4 fiducial probabilities.

Representing ensembles

$$P = \{p_i, \psi_i\}_i$$

$$p(a|P) = \sum_i p_i p(a|\psi_i) = \sum_i p_i \sum_j c_j p(a_j|\psi_i)$$

$$\sum_i p_i \vec{e}_a \cdot \vec{s}_{\psi_i} = \vec{e}_a \cdot \vec{s}_P$$

$$\vec{s}_P = \sum_i p_i \vec{s}_{\psi_i}$$

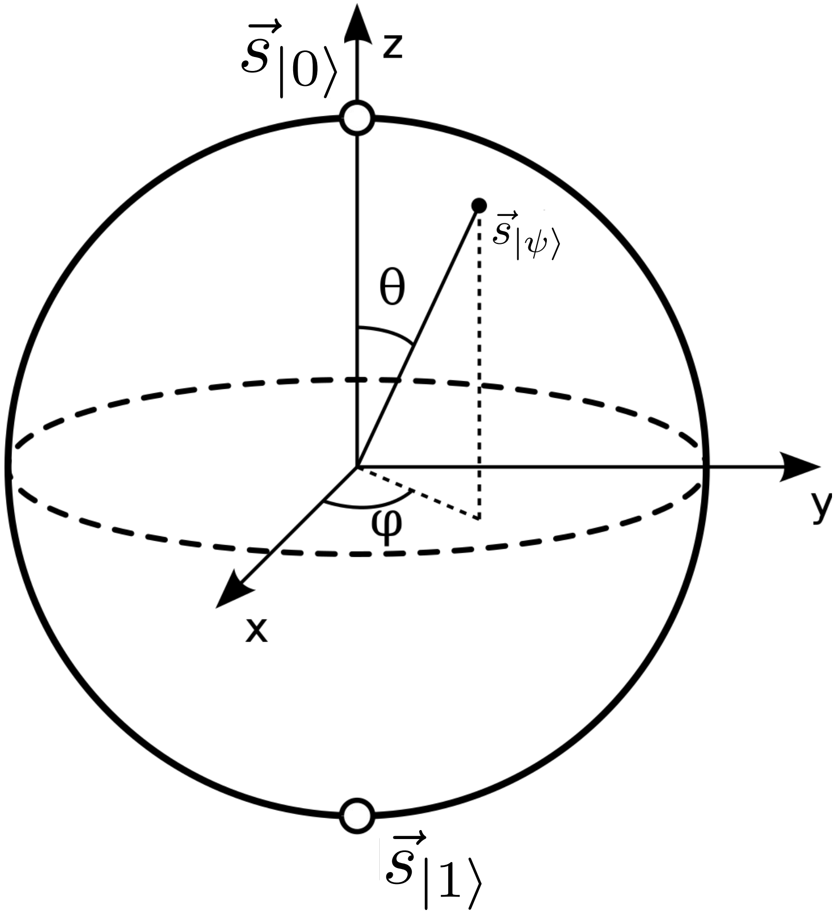
$\vec{s}_P$ is the mixed state representing the ensemble P

Representing ensembles

- Mixed state representing ensemble $P = \{p_i, \psi_i\}_i$ is obtained by taking convex combination of vectors $\vec{s}_{\psi_i}$

$$\vec{s}_P = \sum_i p_i \vec{s}_{\psi_i}$$

Qubit state space



https://commons.wikimedia.org/w/index.php?curid=5829358

Probabilistic representation

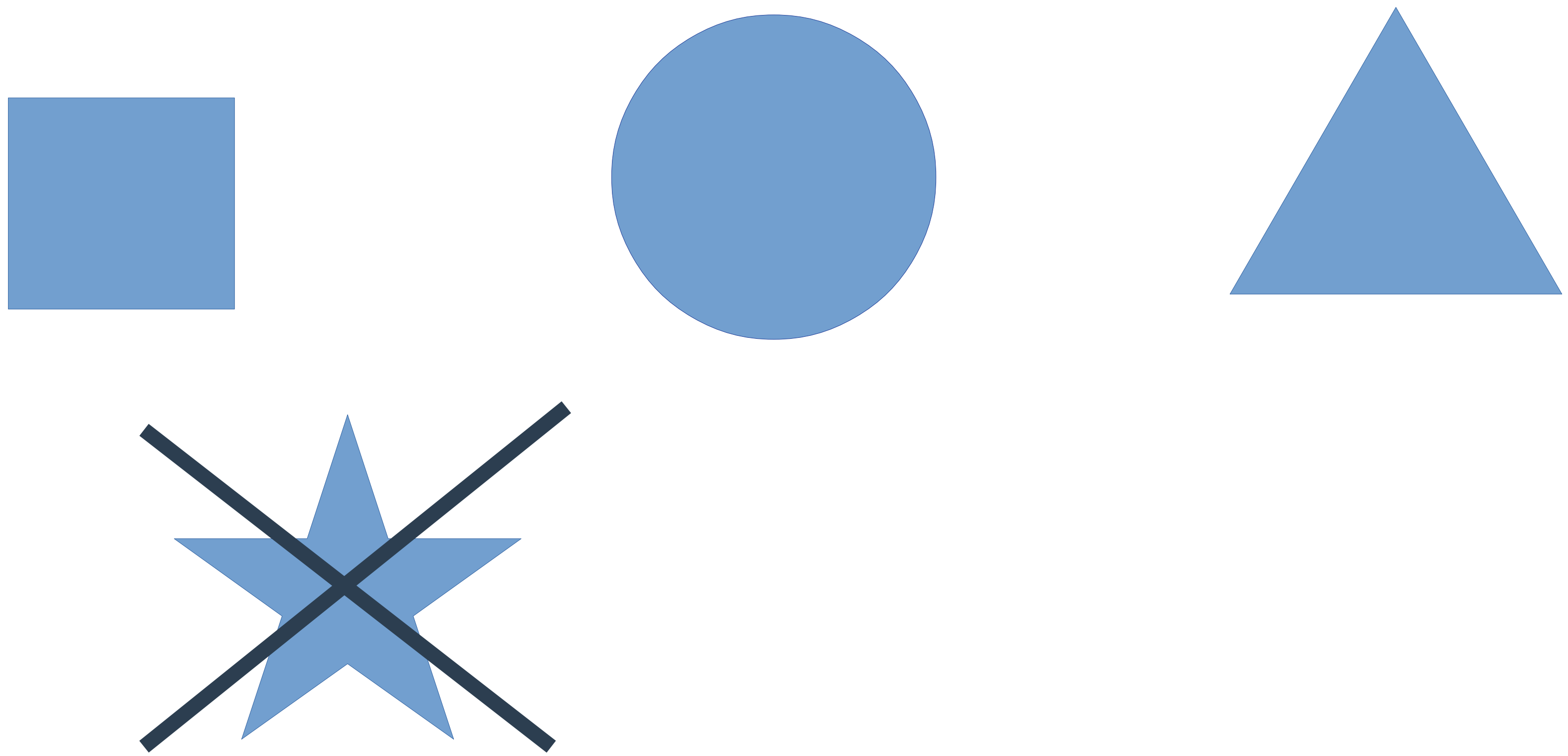
- This is the representation of states in which ensembles are represented by convex combinations of state vectors.
- For example the Bloch vector representation/density matrix representation.
- This is the representation we use in GPTs.

Key facts about the probabilistic representation

- Outcome probabilities are linear functions of the states.
- We went from $p(a|\psi) = |\langle a|\psi\rangle|^2$ to $p(a|\psi) = \vec{e}_a \cdot \vec{s}_\psi$
- We can take mixtures of states by taking convex combinations of state vectors.
- Since for any pair of states we can take a mixture the state space is convex.

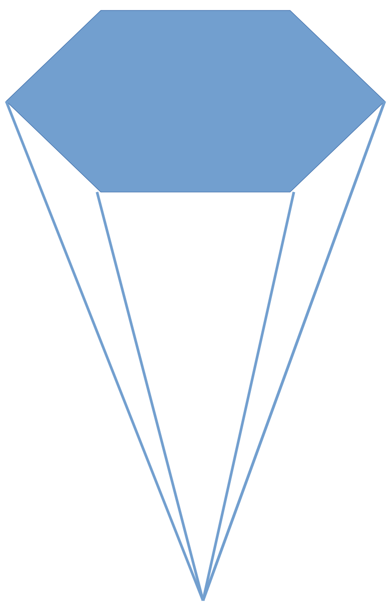
Convex sets

- For every pair of points in the set the line segment joining them is part of the set.

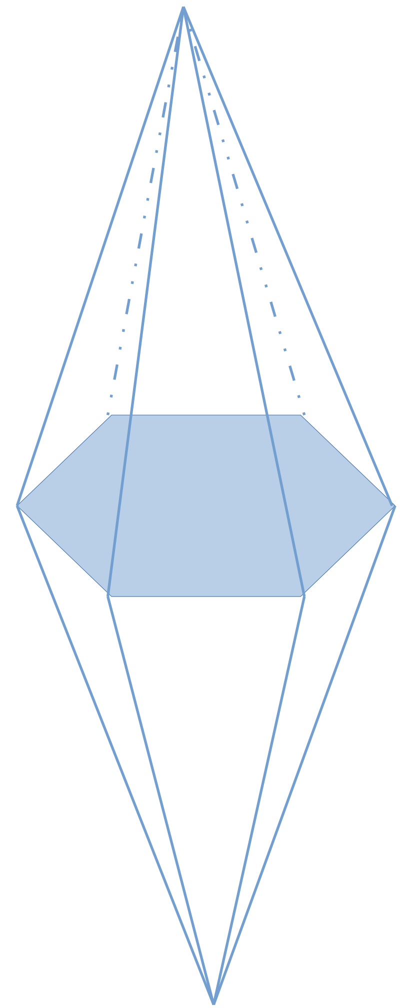


- Classical and quantum state spaces correspond to convex sets.
- Other convex sets correspond to other non-classical systems.
- Like the general lattice structure and non-classical logics.

General GPT: single system



State space $S \subset V_A$



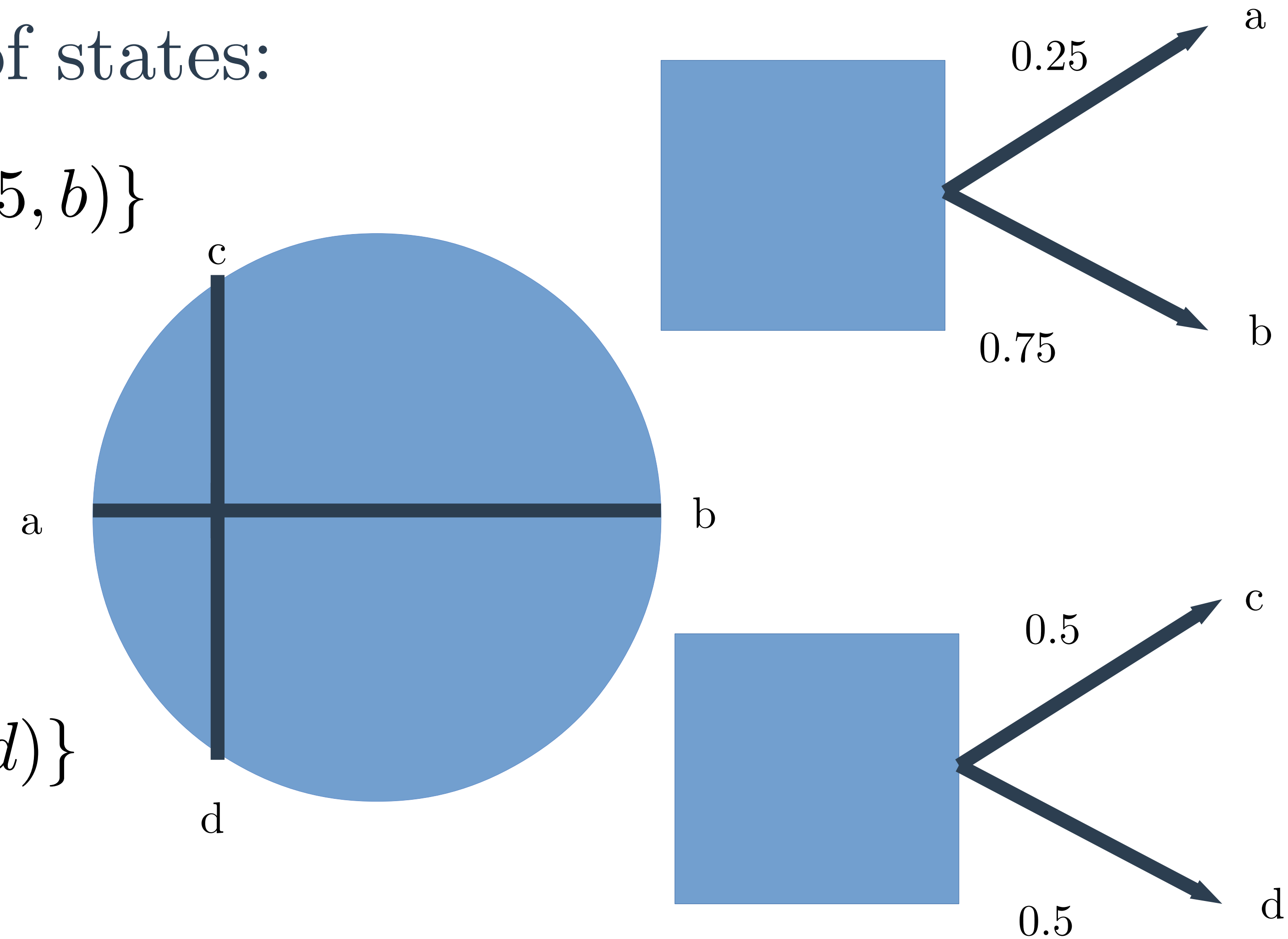
Effect space

$E \subset V_A^*$

How to “read” a convex state space

Mixtures of states:

$$\{(0.25, a), (0.75, b)\}$$

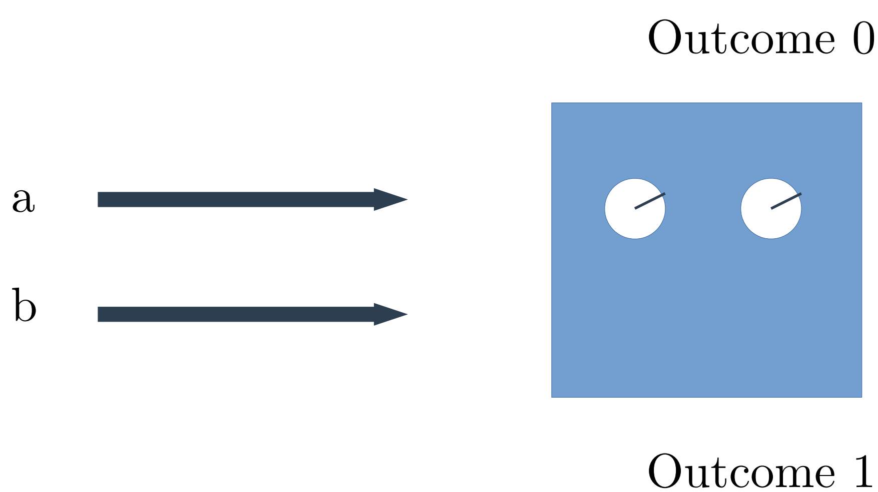


$$\{(0.5, c), (0.5, d)\}$$

“True” definition of mixed states

- Multiple ensembles are assigned the same mixed state.
- A mixed state is an equivalence class of ensembles.
- See Holevo: “Probabilistic and Statistical Aspects of Quantum Theory”

Perfectly distinguishable states



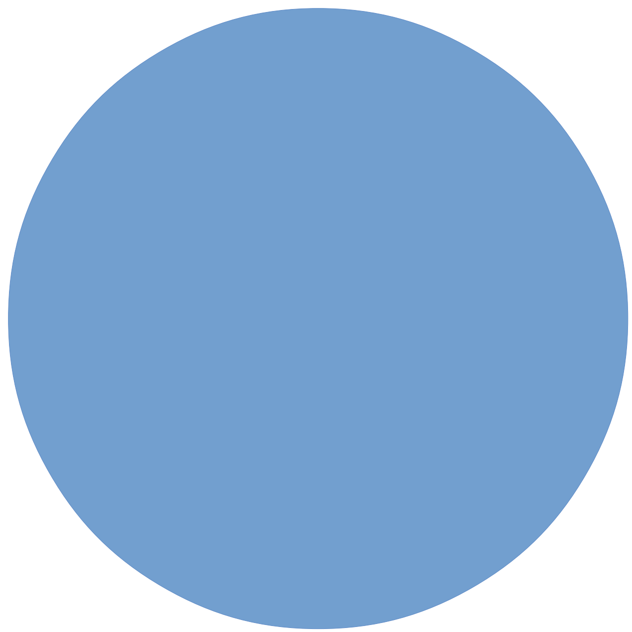
$$p(0|a) = 1$$
$$p(0|b) = 0$$

$$p(1|b) = 1$$
$$p(1|a) = 0$$

How to "read" a convex state space

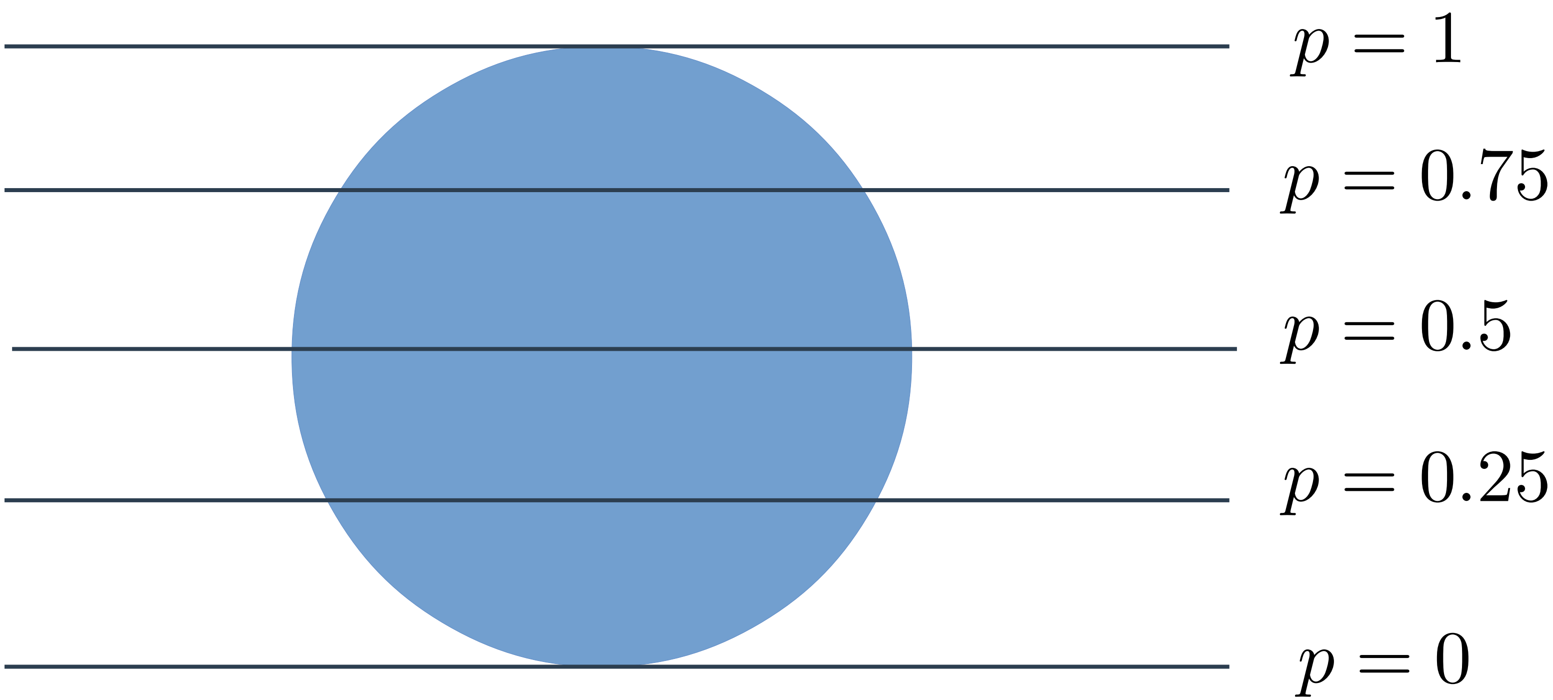
- Effects are linear functionals.

$$p(a|\psi) = \vec{e}_a \cdot \vec{s}_\psi$$



How to “read” a convex state space

- Effects are linear functionals.



Perfectly distinguishable states

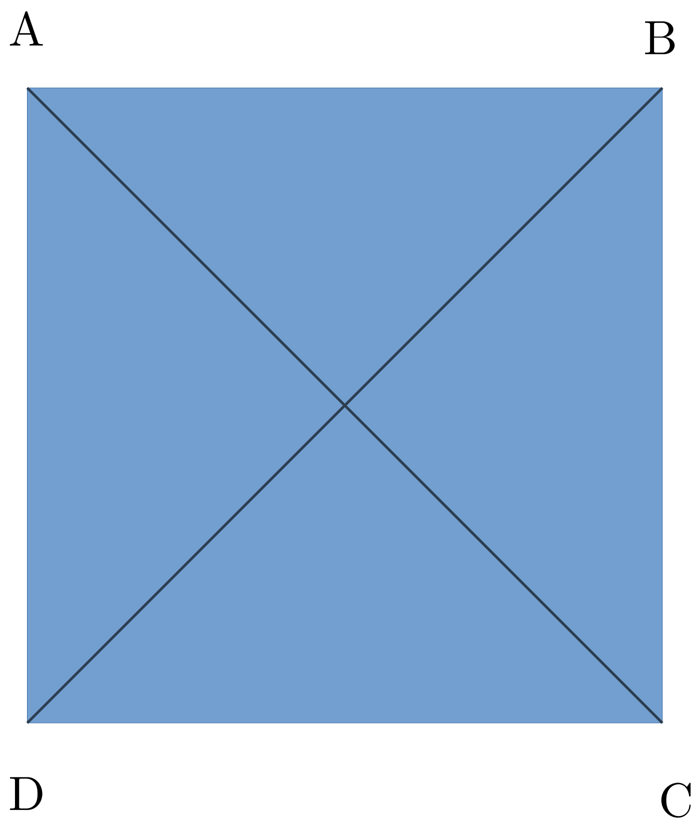
- So antipodal states are perfectly distinguishable.
- There is a single measurement where one of the outcomes occurs with certainty on one of the states, and with probability 0 for the other

The gbit (one half of a PR box)

- Let us consider a system which can be prepared in 4 pure states, and mixtures of these 4 pure states.
- Its state space is the square.
- Like all non-classical state spaces, there are certain ensembles which are indistinguishable.

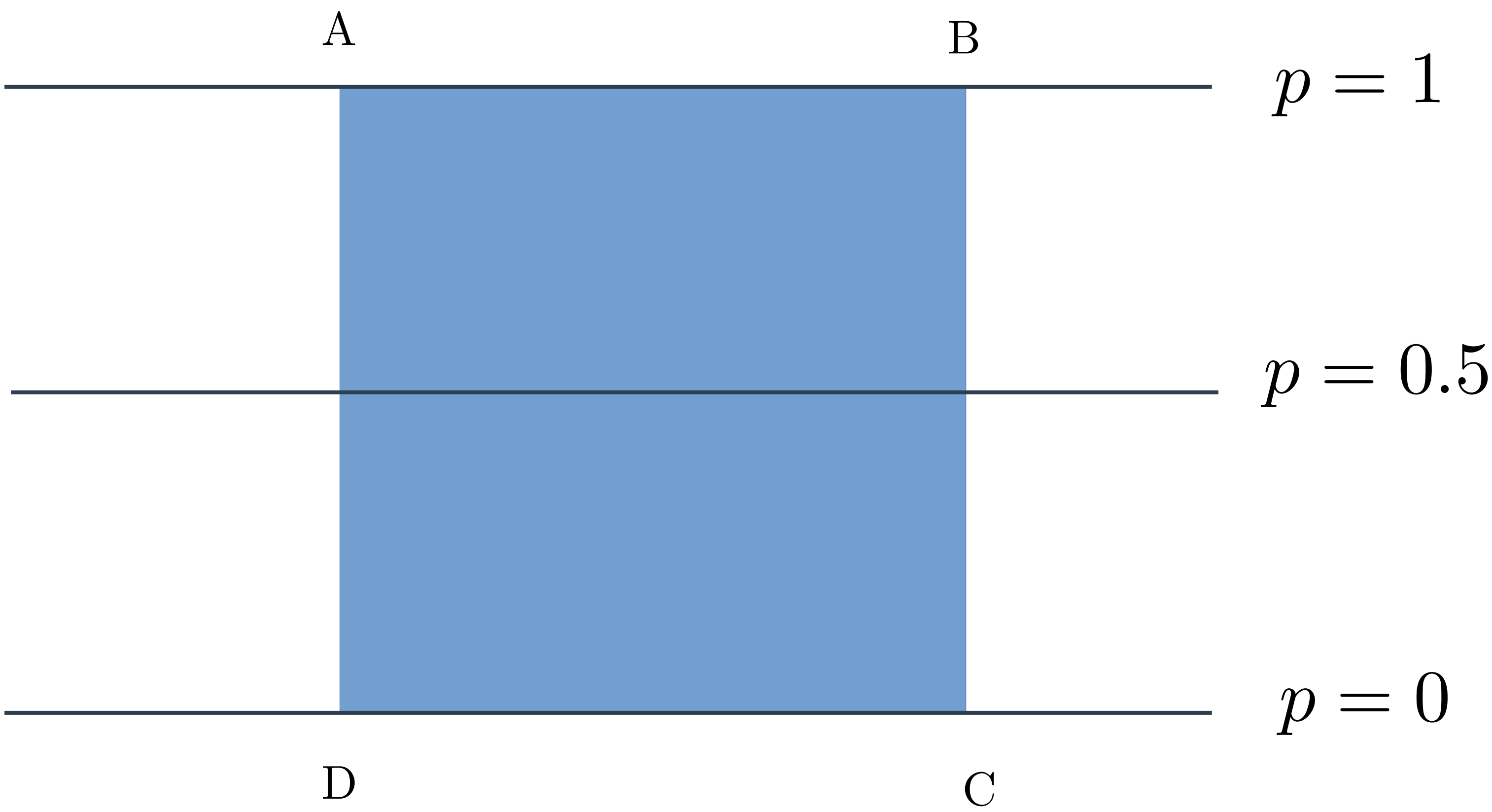
Indistinguishable ensembles

$$\{(\frac{1}{2}, B), ((\frac{1}{2}, D)\}$$

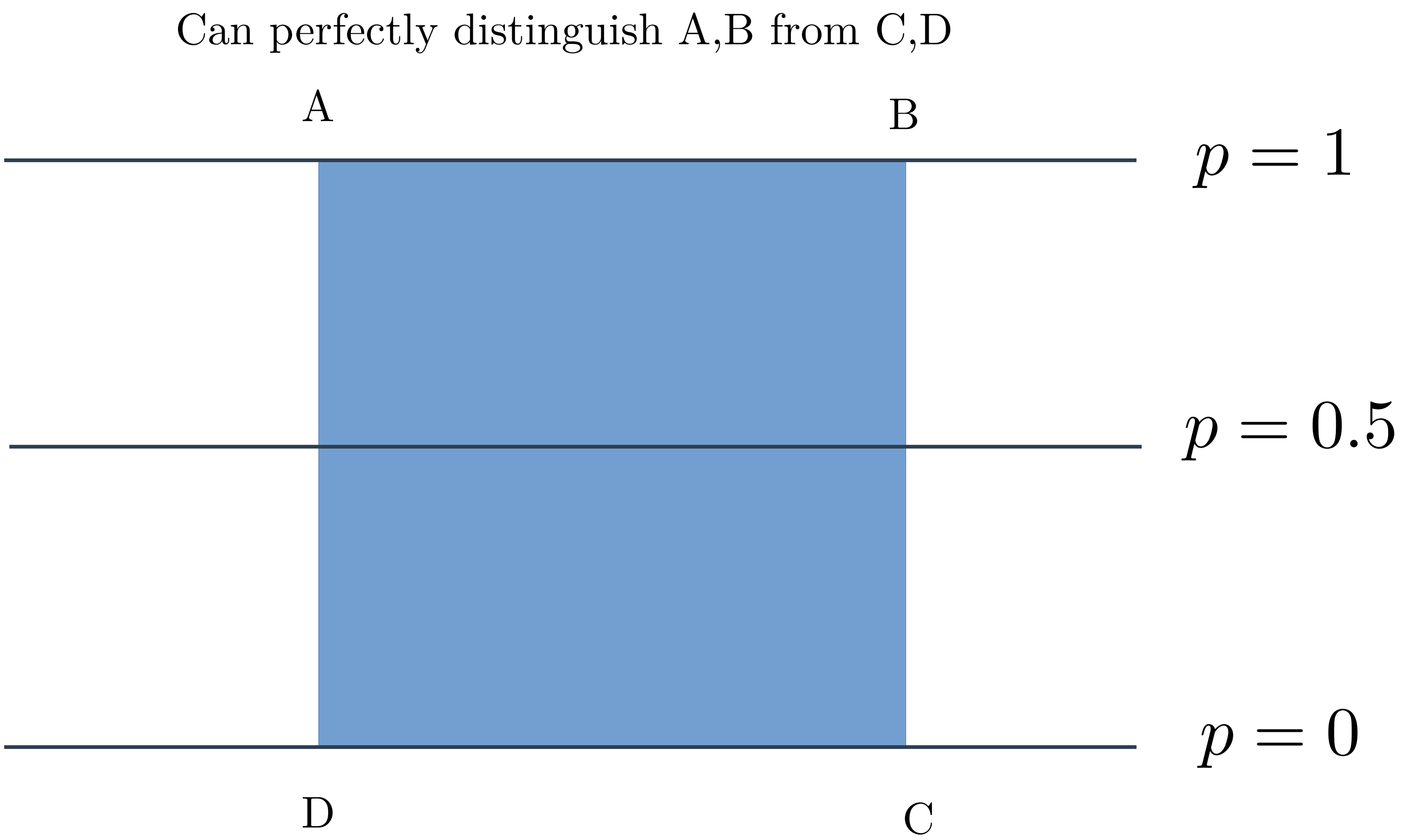


$$\{(\frac{1}{2}, A), ((\frac{1}{2}, C)\}$$

How to "read" a convex state space

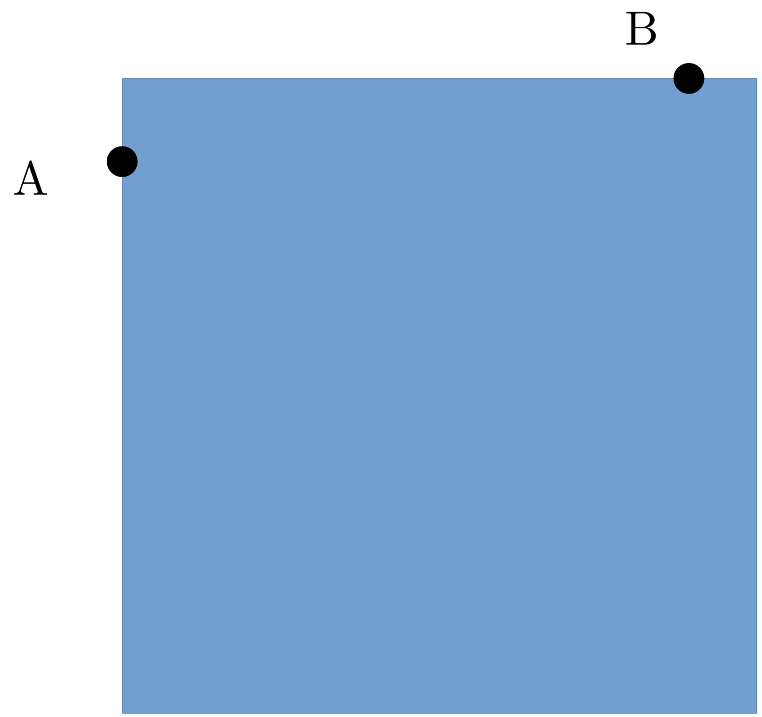


How to "read" a convex state space



How to "read" a convex state space

Can we perfectly distinguish A from B?



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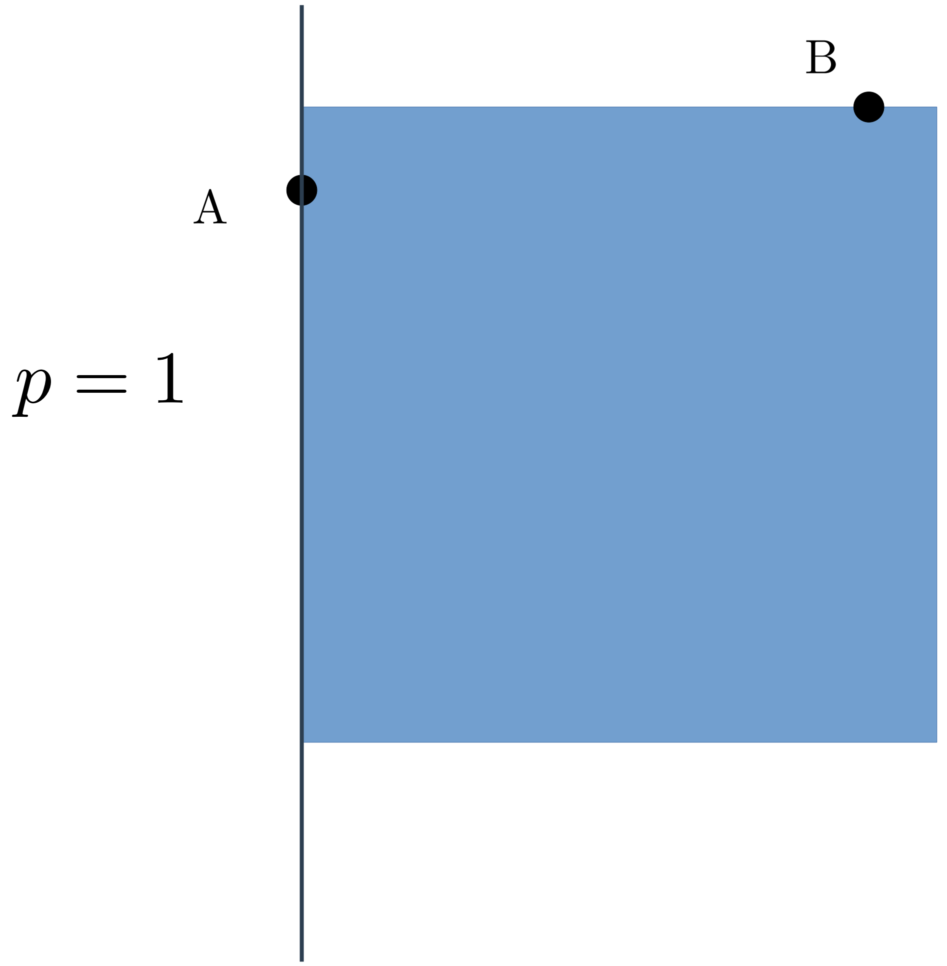
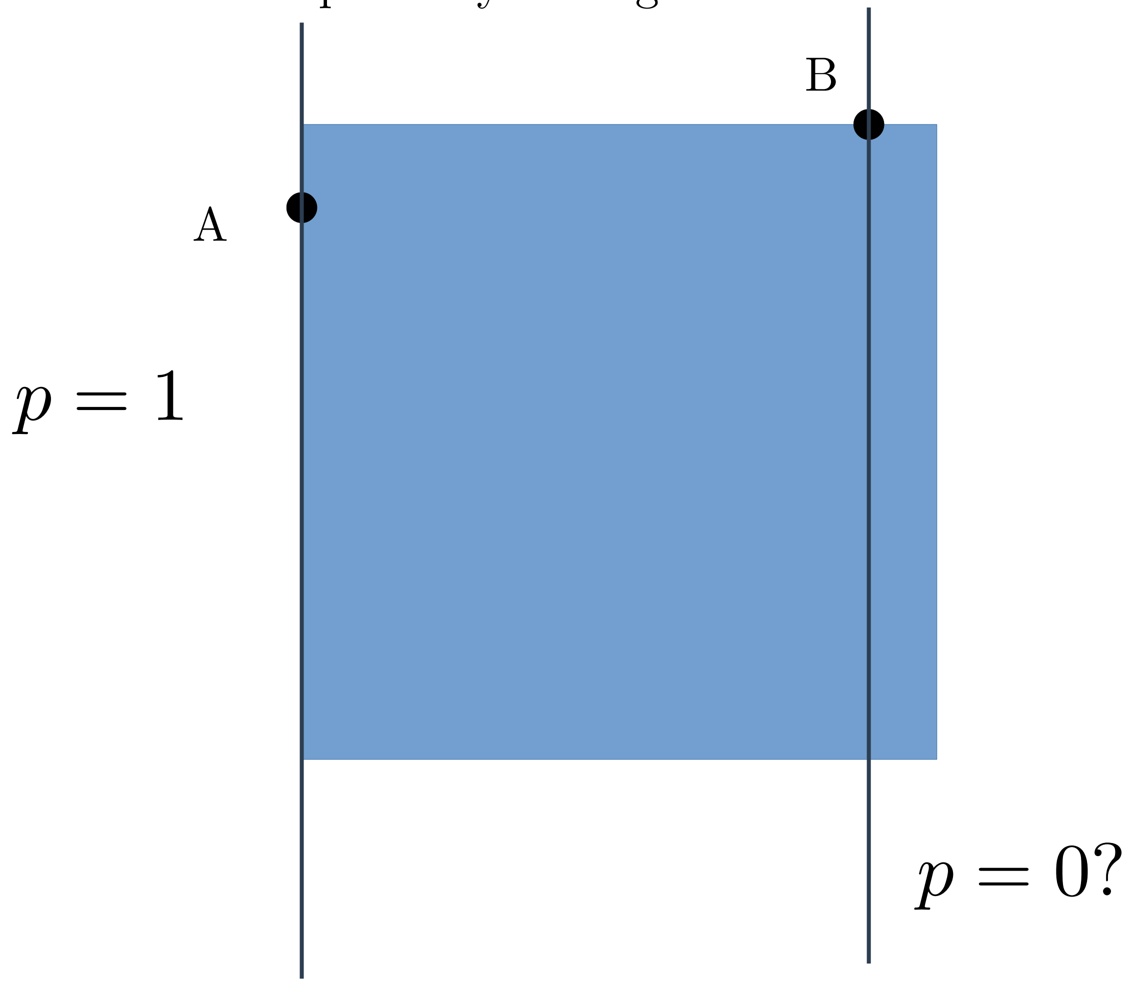


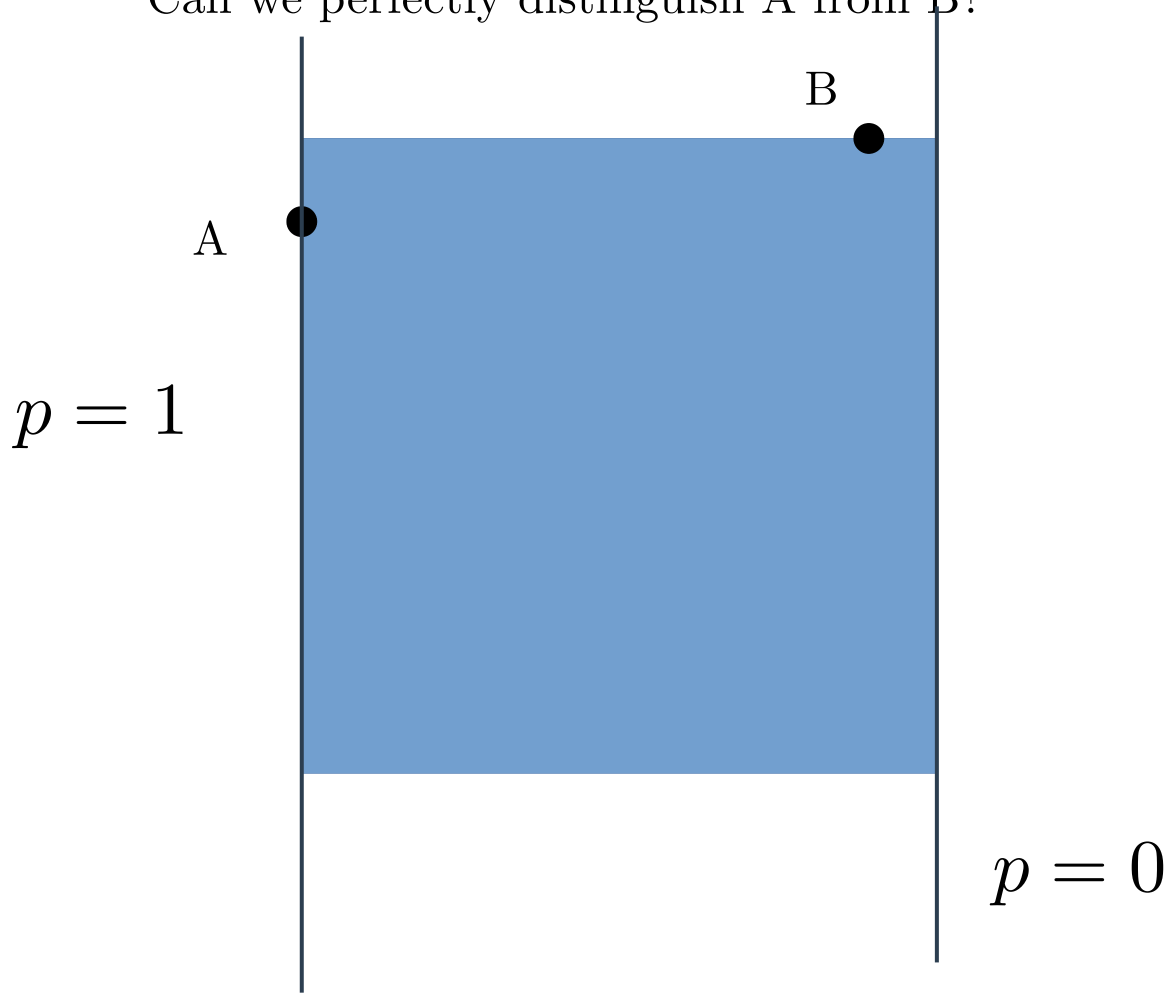
How to "read" a convex state space

Can we perfectly distinguish A from B?



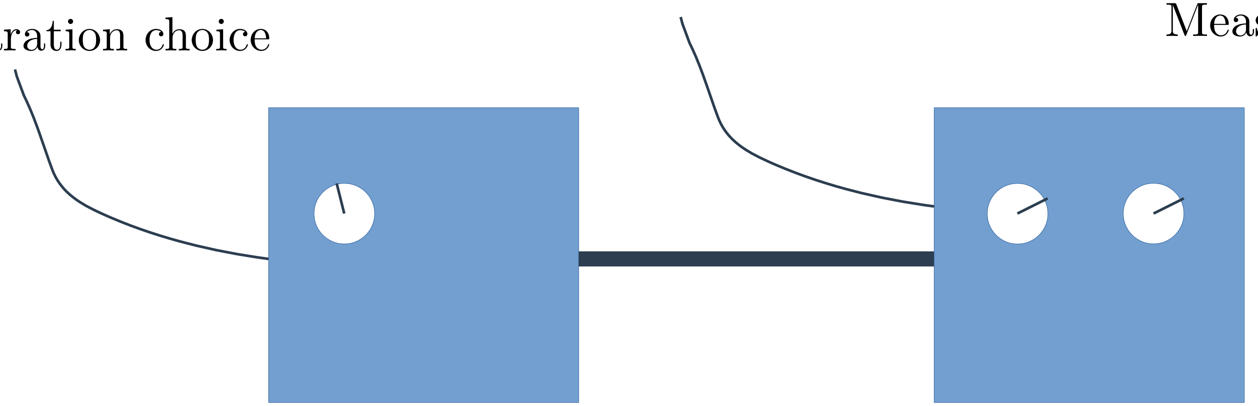
How to "read" a convex state space

Can we perfectly distinguish A from B?



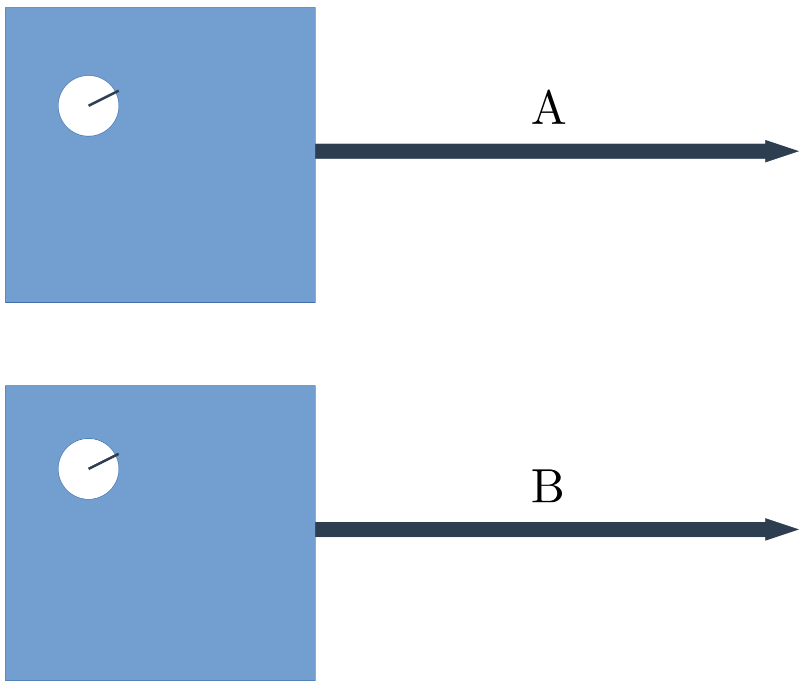
Composite systems

- As well as composing devices in sequence



- Can compose the in parallel

Parallel composition



Operational probabilistic theories

PHYSICAL REVIEW A **81**, 062348 (2010)

Probabilistic theories with purification

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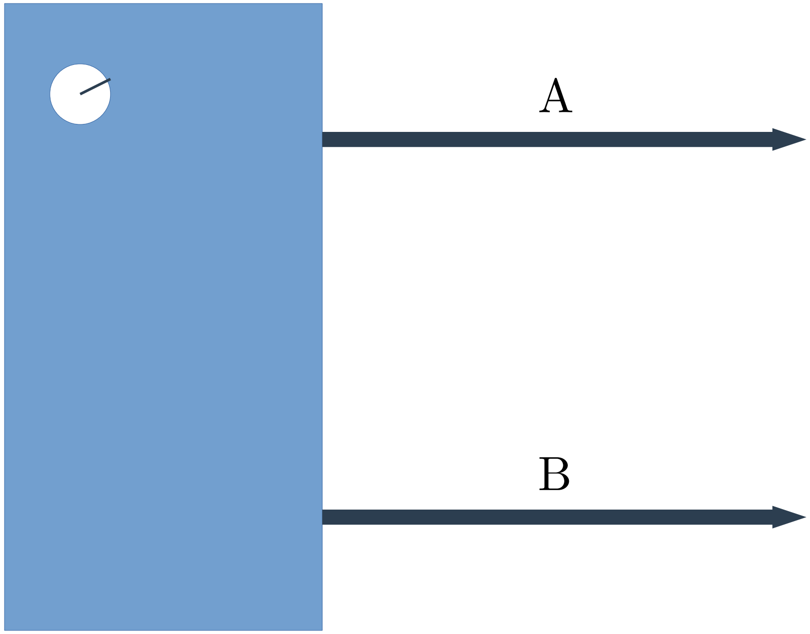
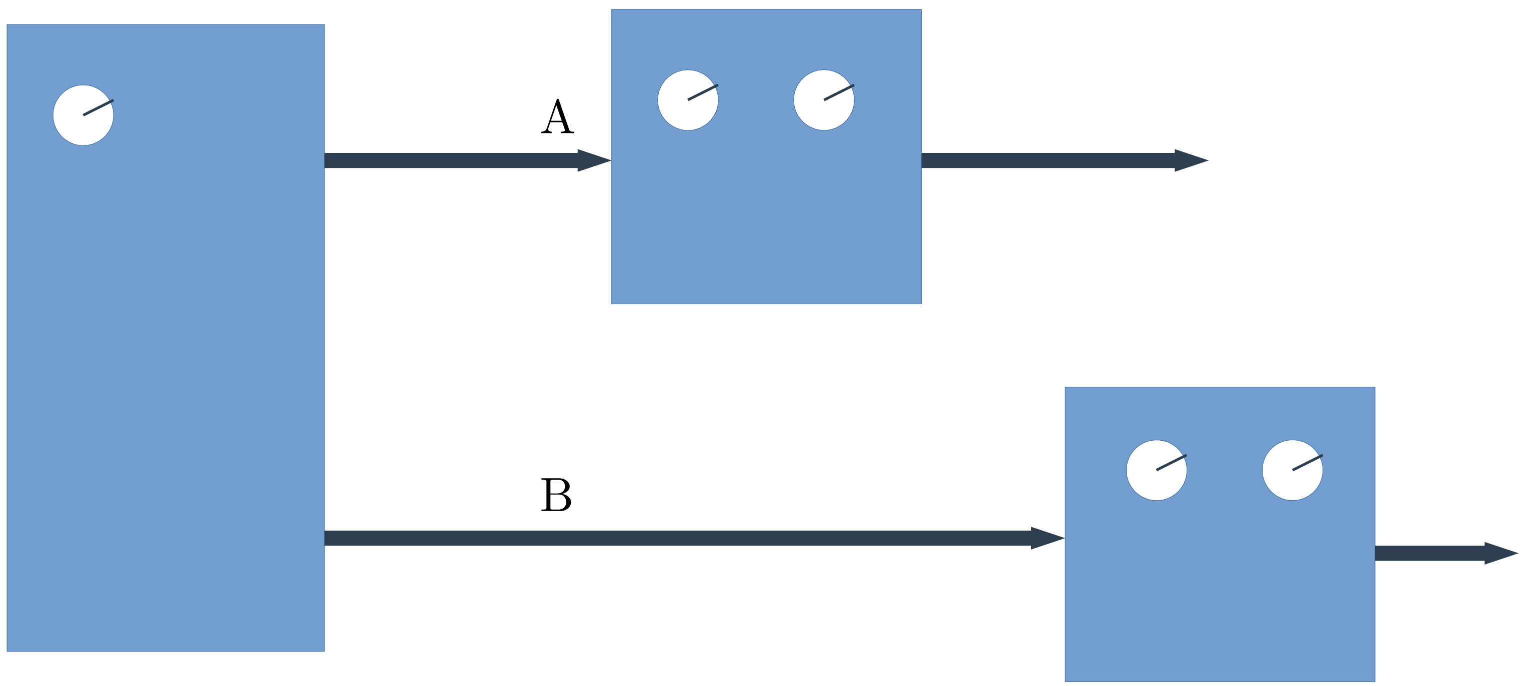
(Received 13 October 2009; published 30 June 2010)

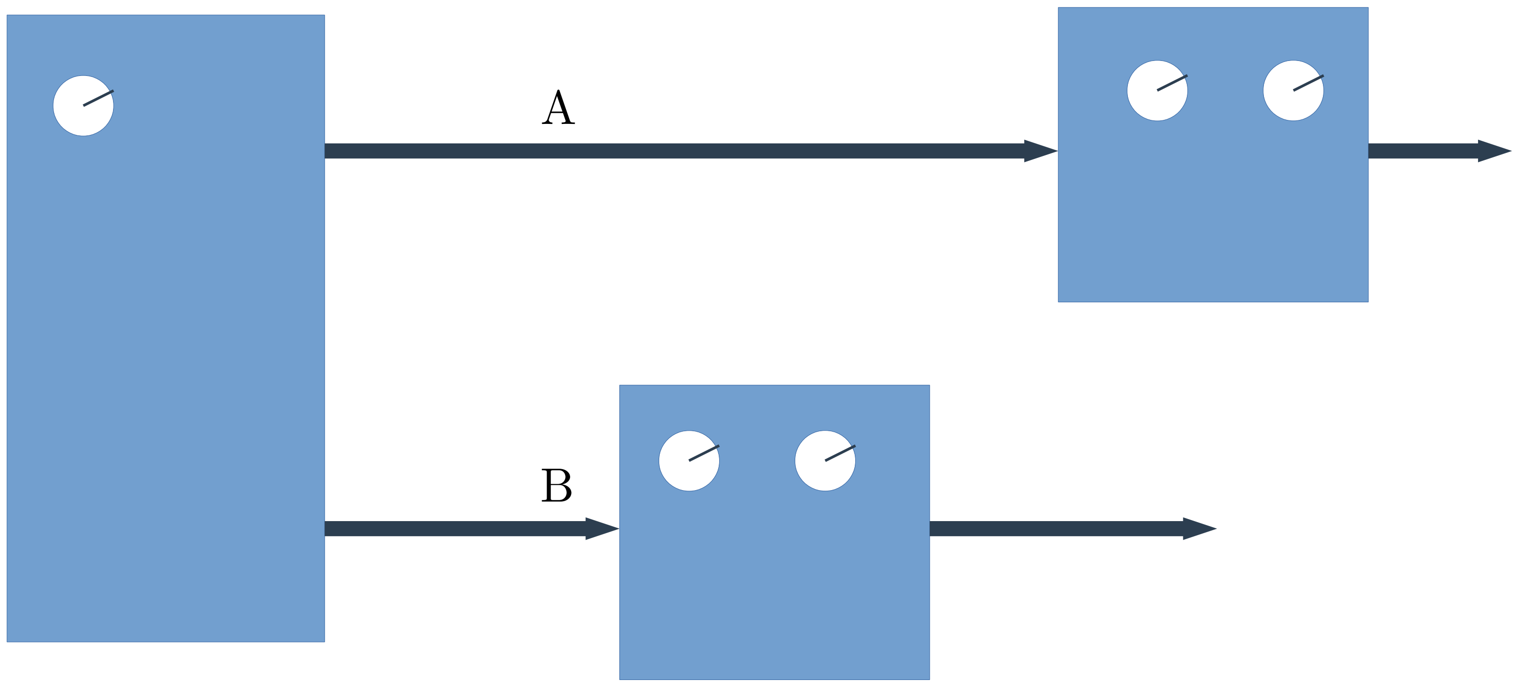
Composite systems



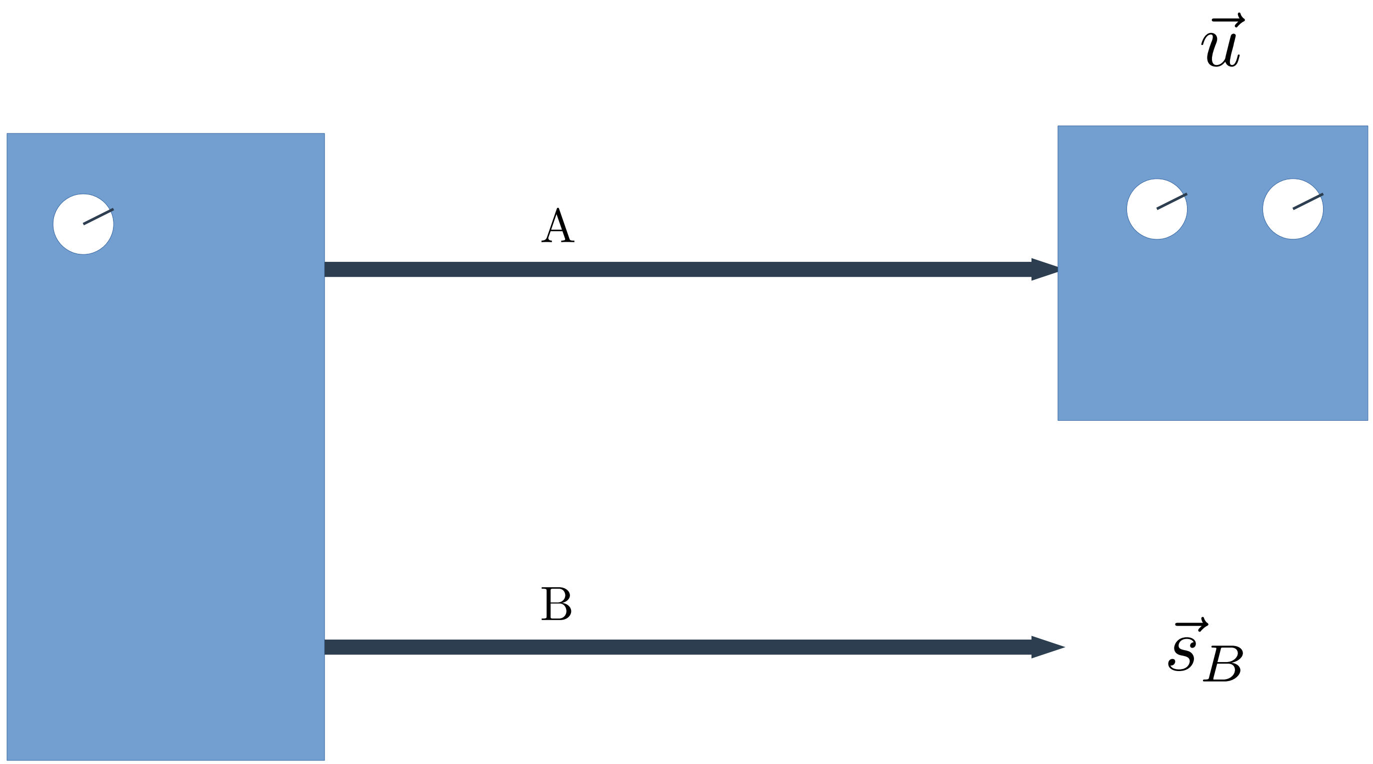
Composite systems (no-signalling)



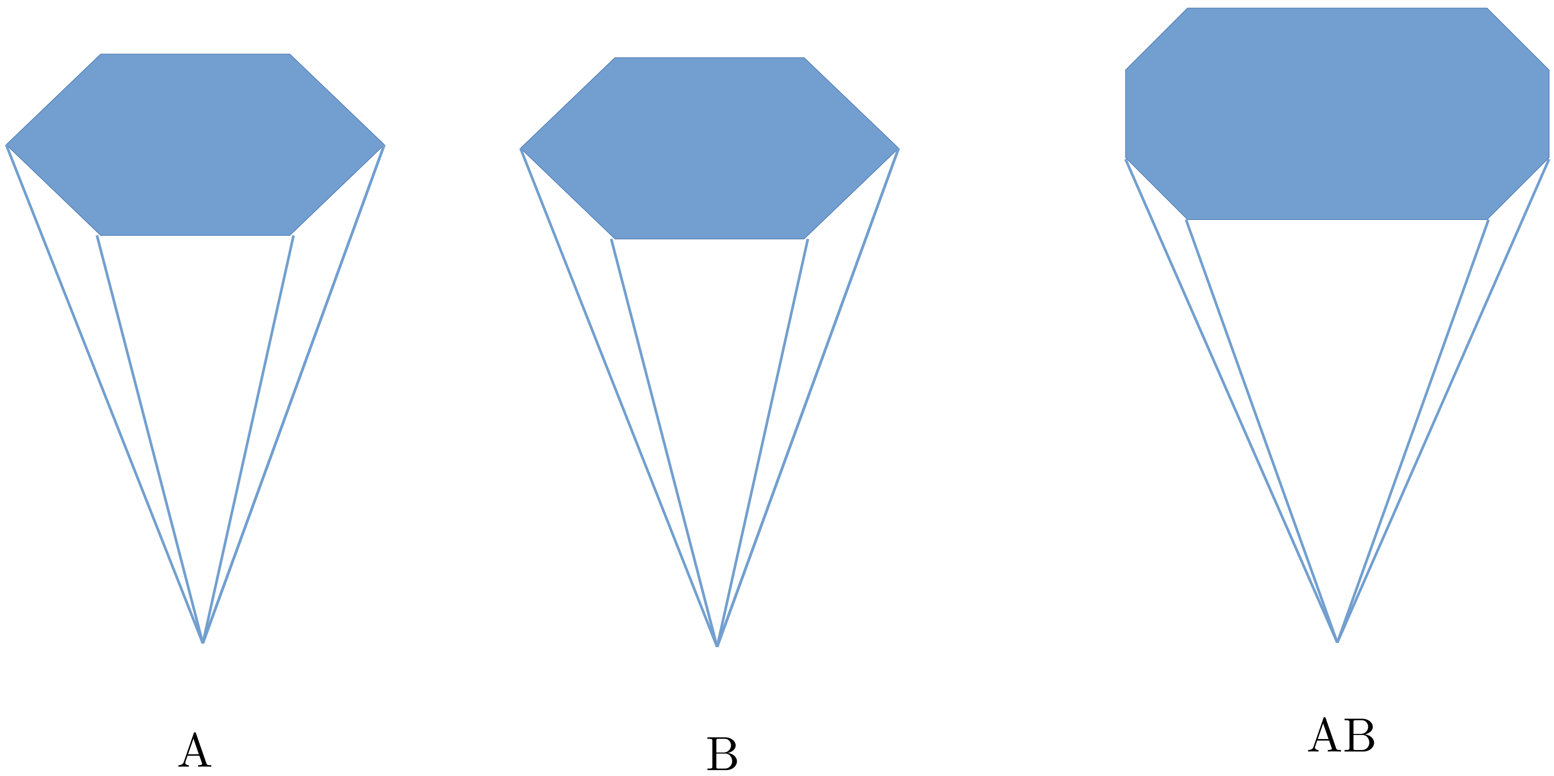
Composite systems (no-signalling)



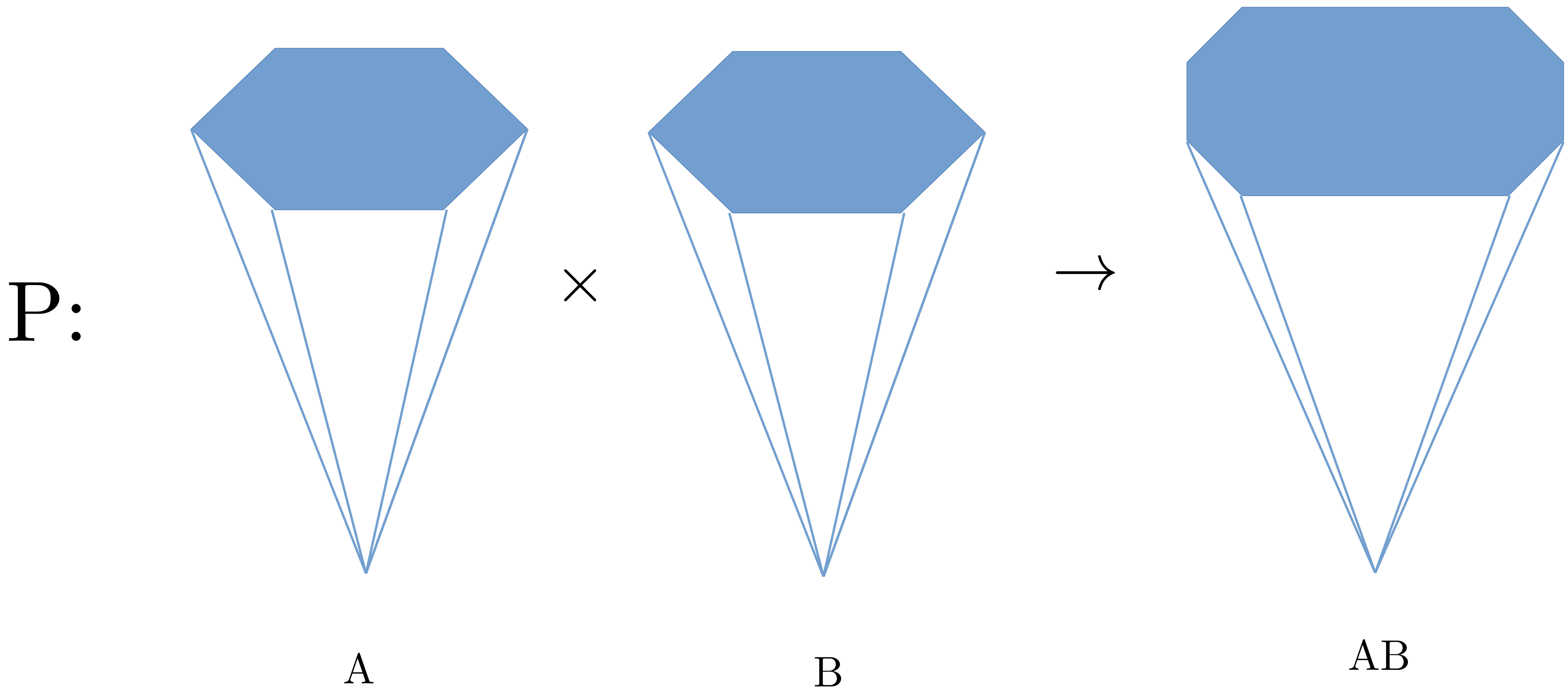
Reduced states



Composite systems: state spaces



Product states



Product states: quantum theory

$$P_{quantum} : s_\psi \times s_\phi \mapsto s_\psi \otimes s_\phi$$

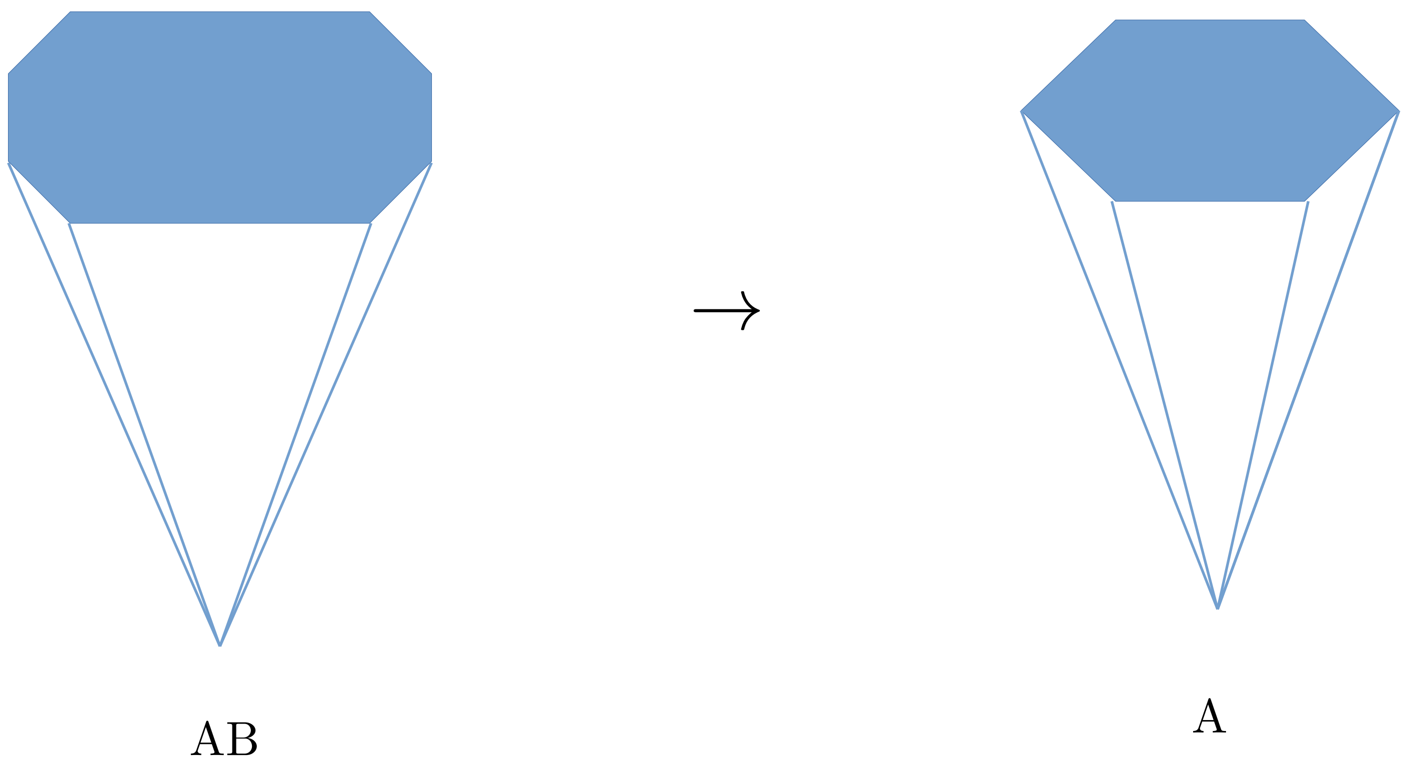
$$P_{quantum} : \rho \times \sigma \mapsto \rho \otimes \sigma$$

Product states

- This map is bilinear (due to mixing)
- If we assume local tomography it is just a tensor product
- But not in general

Reduced states

R:



Reduced states: quantum theory

$$R_{\text{quantum}} : \rho_{AB} \mapsto \text{Tr}_B(\rho_{AB})$$

Reduced states

- This map is a linear map from V_{AB} to V_A
- Obtained by taking the unit effect on subsytem B and the identity on A
- $\mathbb{I}_A \star u_B$

Effect space

- We have equivalent maps on the effect space.

Conclusion

- Have a framework within which we can talk about operational theories: general probabilistic theories
- Highlighted the convex structure, which arises from the possibility of taking mixtures
- Mentioned the categorical structure which arises from composing devices
- Arbitrary convex sets correspond to non-classical systems

Redundancy of the measurement postulates of quantum theory

MENU

nature COMMUNICATIONS

Article | OPEN | Published: 25 March 2019

The measurement postulates of quantum mechanics are operationally redundant

Lluís Masanes, Thomas D. Galley & Markus P. Müller

Nature Communications **10**, Article number: 1361 (2019) | Download Citation

Quantum theory

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements $(F_1, ..., F_n) \quad F_i \geq 0 \ \sum_i F_i = \mathbb{I}$

4) Probabilities $p(F_i|\psi) = \langle\psi|F_i|\psi\rangle$

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements $(F_1, ..., F_n)$ $F_i \geq 0$ $\sum_i F_i = \mathbb{I}$

4) Probabilities $p(F_i|\psi) = \langle\psi|F_i|\psi\rangle$

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Alternative probabilistic structure

Outcome Probability Function

$$P(F|\psi) = F(\psi)$$

$$F : \mathrm{P}\mathbb{C}^d \to [0, 1]$$

Alternative probabilistic structure

Measurement

$$(F_1, ..., F_n)$$

$$\sum_i F_i(\psi) = 1 \quad \forall \psi \in \mathrm{P}\mathbb{C}^d$$

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements $(F_1, ..., F_n) \quad \sum_i P(F_i|\psi) = 1 \ \forall \psi$

4) Probabilities $p(F|\psi) = F(\psi)$

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Theories with modified measurements

- A measurement postulate is a set of OPFs
- $\mathcal{F}_d$ for every $d \geq 2$
- Any subset which sum to 1 form a measurement
- In the case of quantum theory we have

$$\mathcal{F}_d = \{\langle\psi|F|\psi\rangle|\forall F \text{ s.t. } 0 \leq F \leq \mathbb{I}\}$$

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements

$$\mathcal{F}_d$$

4) Probabilities

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements

$$\mathcal{F}_d$$

Probabilistic structure

4) Probabilities

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements

$$\mathcal{F}_d$$

4) Probabilities

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Dynamical Structure

Probabilistic structure

Quantum theory

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3)

$$\mathcal{F}_d = \{F(\psi) | F(\psi) = \langle\psi|\hat{F}|\psi\rangle,\ \hat{F} \geq 0\}$$

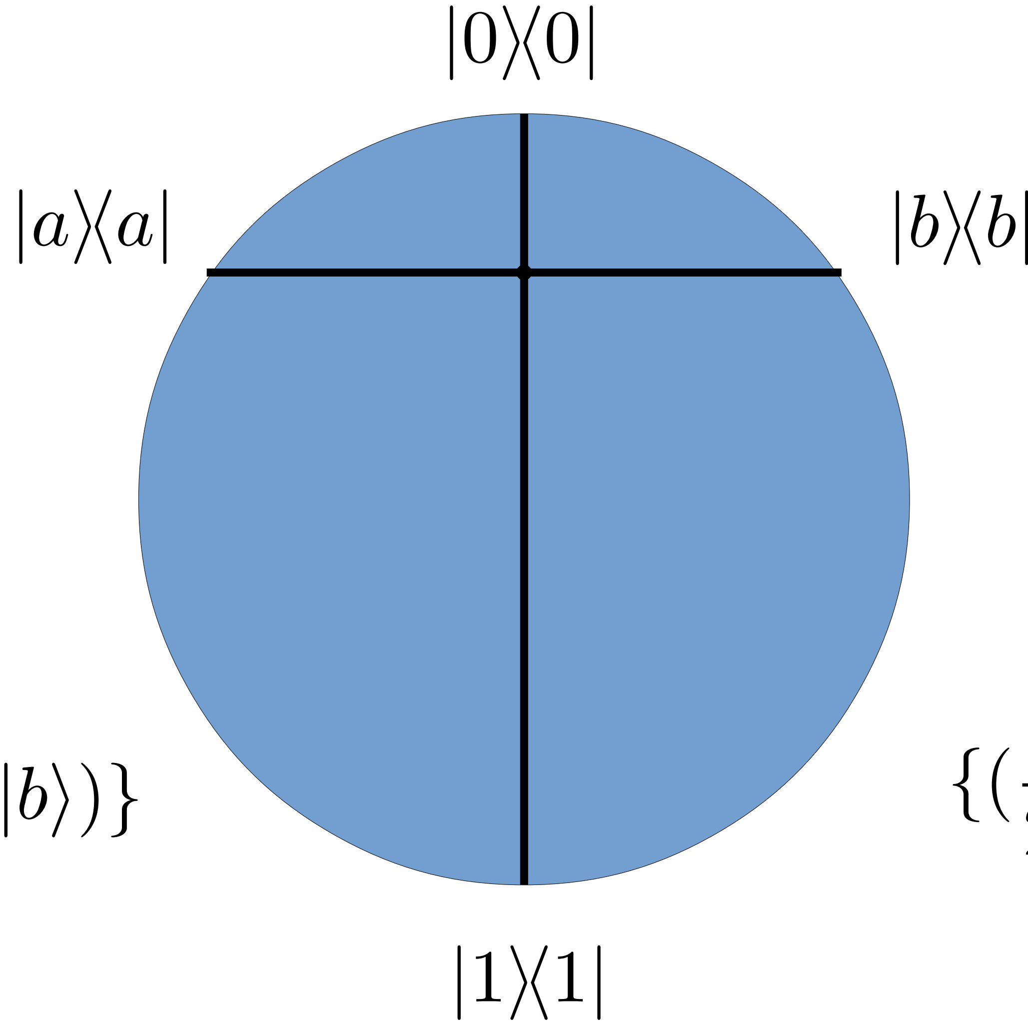
4)

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Warning!

Changing the measurement postulates will change the structure of mixed states

What are mixed states?



$$\{(\frac{1}{2}, |a\rangle), (\frac{1}{2}, |b\rangle)\}$$

$$\{(\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle)\}$$

Mixed states

Mixed states are equivalence classes of indistinguishable ensembles.

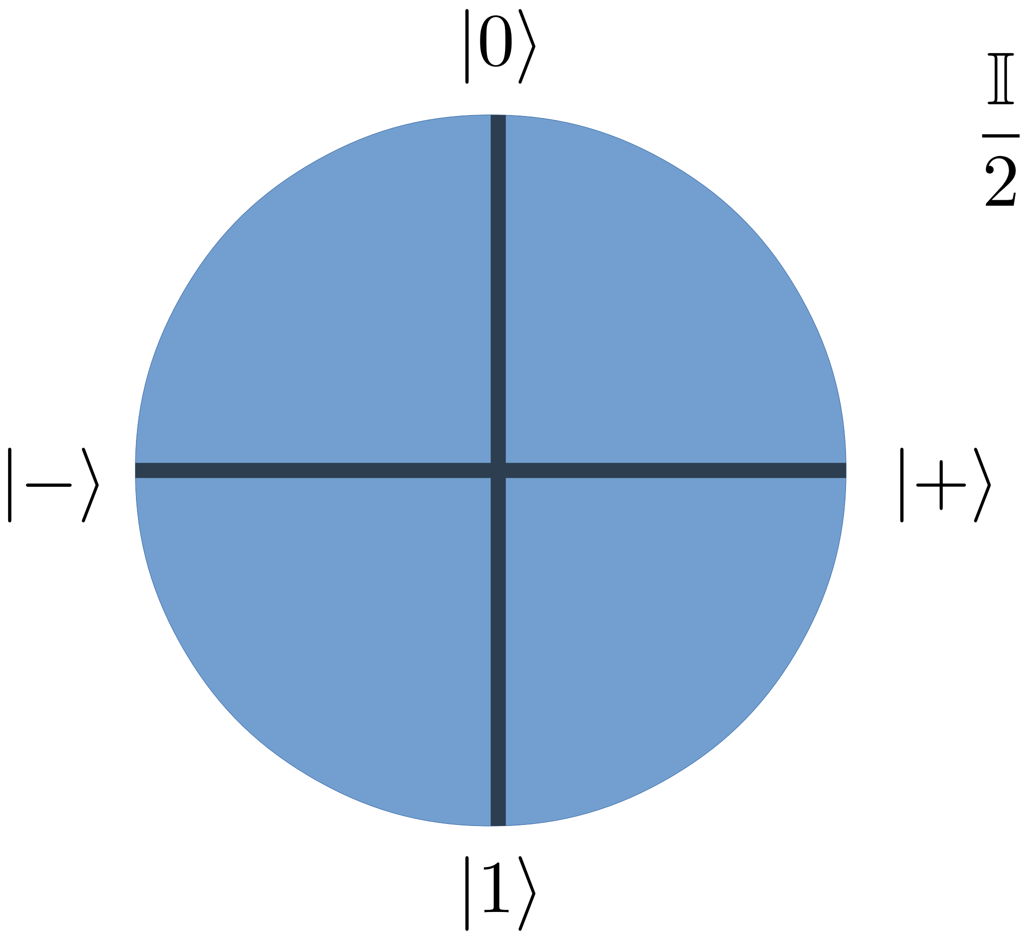
Indistinguishability is relative to the available measurements.

Changing the measurements will change which ensembles are indistinguishable.

State spaces

Systems with alternative measurement postulates have same pure states but different mixed states.

Mixed states: rebit



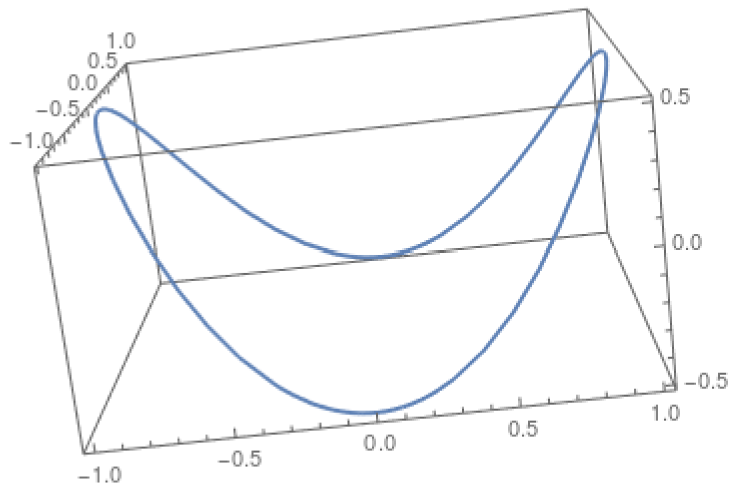
Mixed states

- If we allowed different measurements, these two ensembles may be distinguishable.

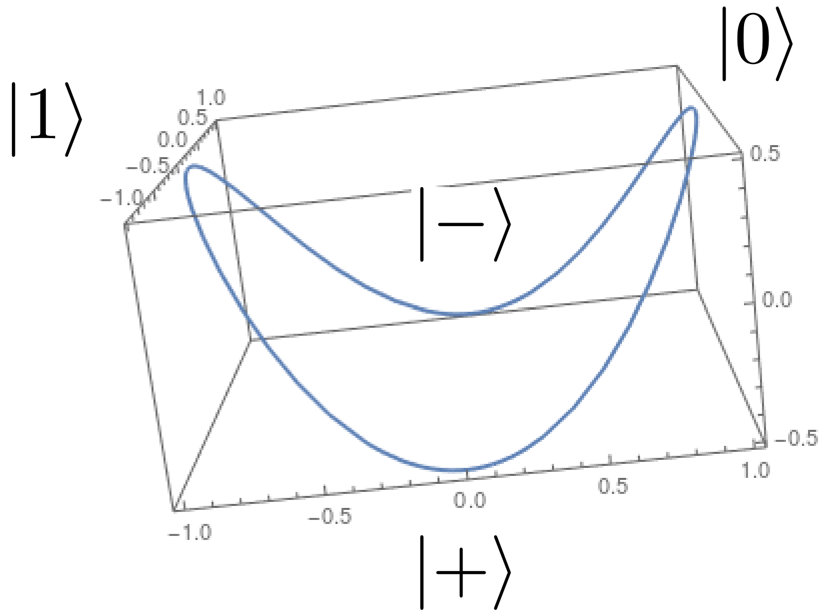
$$\{(\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle)\}$$

$$\{(\frac{1}{2}, |+\rangle), (\frac{1}{2}, |-\rangle)\}$$

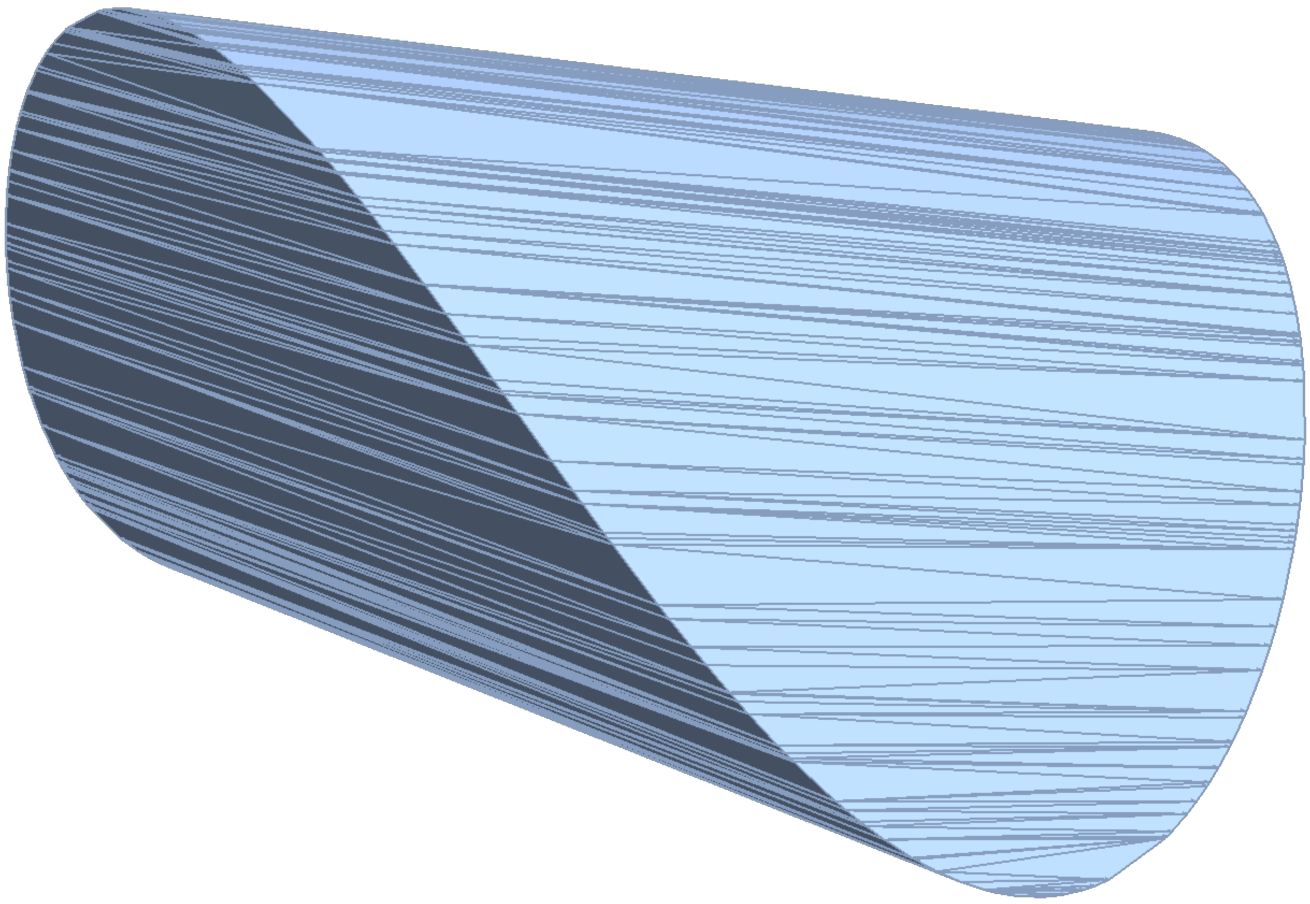
Alternative rebit



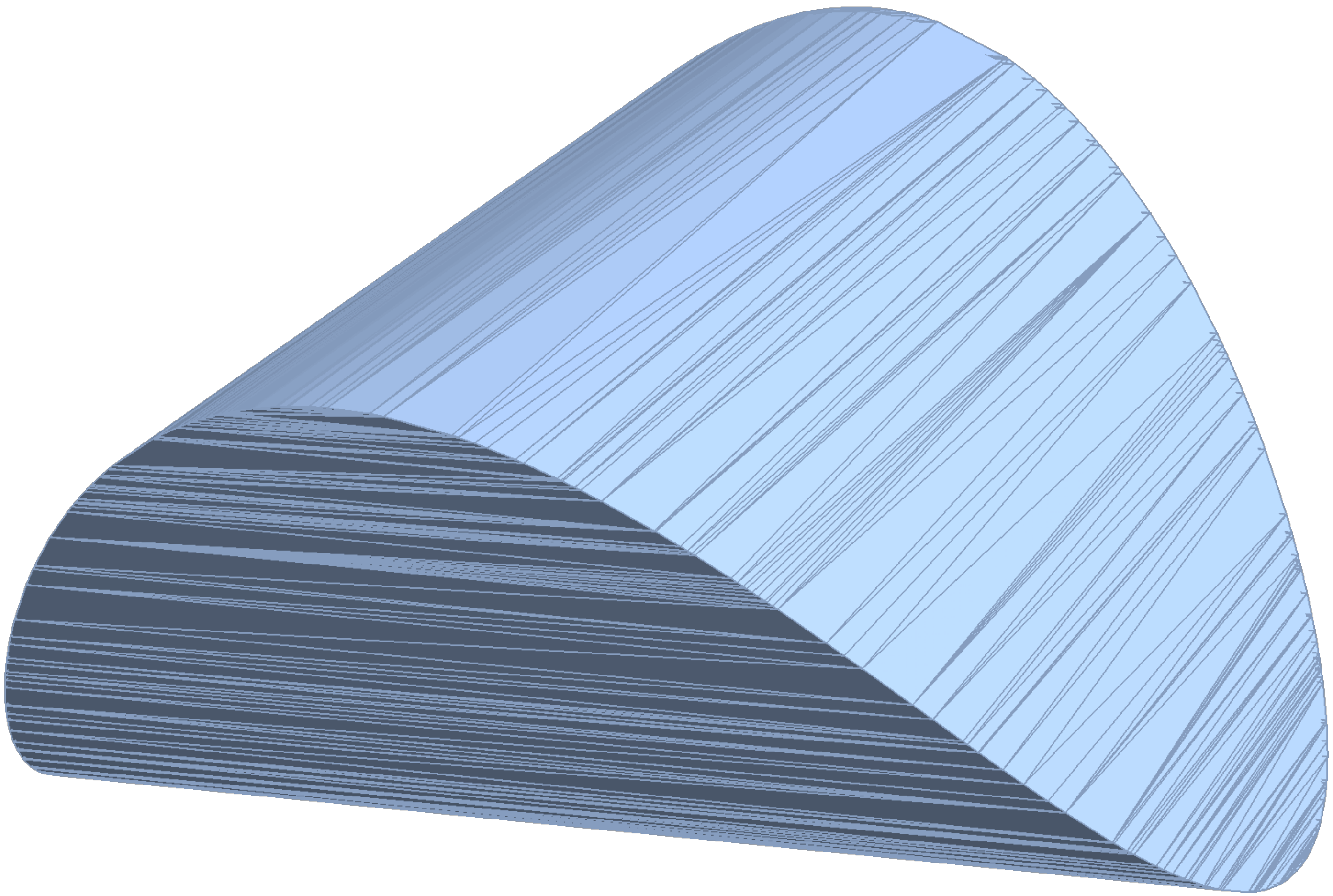
Alternative rebit



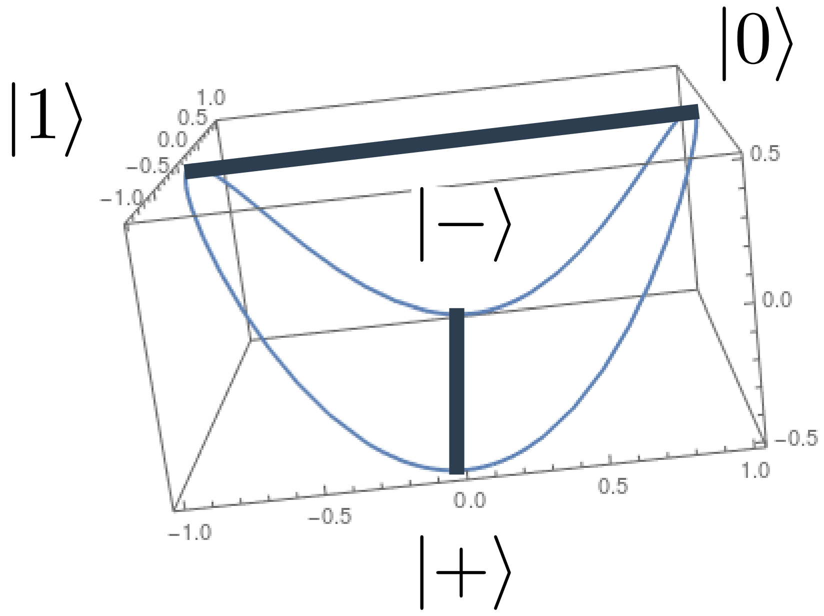
Alternative rebit



Alternative rebit



Alternative rebit



$$\{(\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle)\}$$

$$\{(\frac{1}{2}, |+\rangle), (\frac{1}{2}, |-\rangle)\}$$

Mixed states

- We can no longer represent mixed states by density matrices when we change the measurements.
- Need to re-derive the mixed states

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements

$$\mathcal{F}_d$$

4) Probabilities

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

Deriving the state space

- Find a set of fiducial measurements in $\mathcal{F}_d$

$$f_1, ..., f_n$$

$$f(\psi) = \sum_{i=1}^{n} c_i f_i(\psi)$$

- Quantum theory $n = d^2$

Deriving the state space

$$f(\psi) = \sum_{i=1}^{n} c_i f_i(\psi)$$

$$\vec{e}_f = (c_1, \ldots, c_n)$$

$$\vec{s}_\psi = \begin{pmatrix} f_1(\psi) \\ \vdots \\ f_n(\psi) \end{pmatrix}$$

$$f(\psi) = \vec{e}_f \cdot \vec{s}_\psi$$

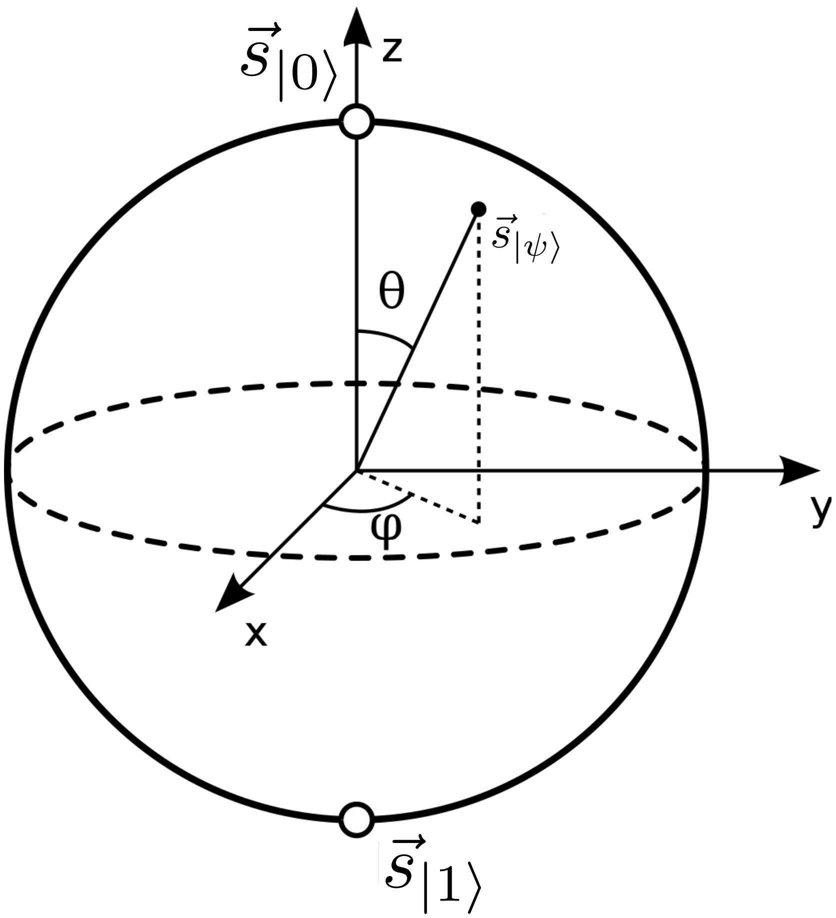
Mixed states

- The pure states of quantum theory correspond to rays in $\psi \in \mathrm{P}\mathbb{C}^d$
- Every embedding of these pure states into gives a system with same pure states but different measurements
- So pure state structure of quantum theory does not fix the measurements.

Classifying all alternatives to the Born rule

- So can classify all alternatives to the measurements postulates
- All state spaces with same pure states but different mixed states as quantum theory
- All possible embeddings of $\mathrm{P}\mathbb{C}^d$ (as a manifold) in $\mathbb{R}^n$

Representation theory and the Bloch sphere



$$\vec{s}_{|\psi\rangle} = \begin{pmatrix} P(+X|\psi) \\ P(+Y|\psi) \\ P(+Z|\psi) \end{pmatrix}$$

$$\vec{s}_{|\psi\rangle} \to R_U \vec{s}_{|\psi\rangle}$$

Probabilistic representation of transformations

$$|\psi\rangle \mapsto \vec{s}_{|\psi\rangle}$$

$$U \mapsto R_U$$

Probabilistic representation of transformations

$$|\psi\rangle \rightarrow \vec{s}_{|\psi\rangle}$$

$$U \rightarrow R_U$$

Representation of SU(d)

$$U_1 U_2 \mapsto R_{U_1} R_{U_2}$$

- Alternative measurement postulate implies different mixed states
- Different representations act on these different mixed states
- Alternative measurement postulate in correspondence with representation of SU(d)

Classification of all alternatives to the Born rule in terms of informational properties

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Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom

June 14, 2017

Theories with modified measurements

1) States $\psi \in \mathrm{P}\mathbb{C}^d$

2) Transformations $\psi \to U\psi, \quad U \in \mathrm{SU}(d)$

3) Measurements

$$\mathcal{F}_d$$

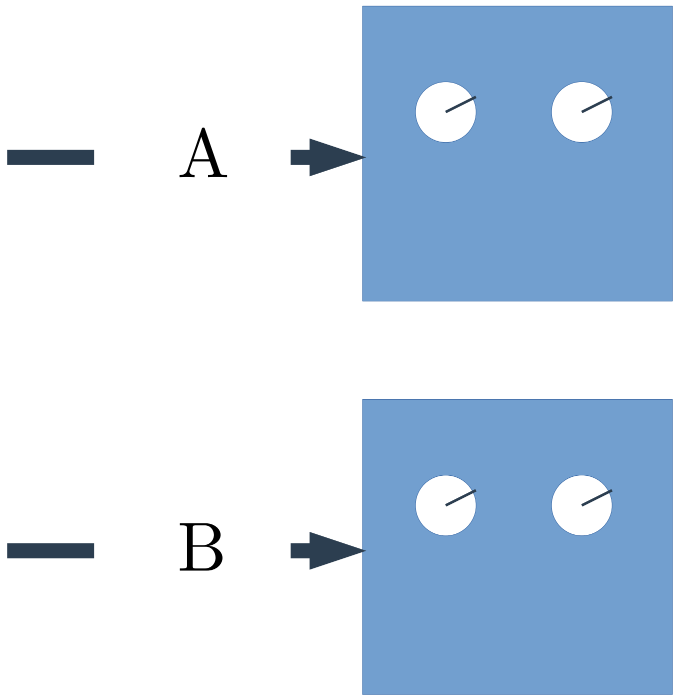
4) Probabilities

5) Composition $\psi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$

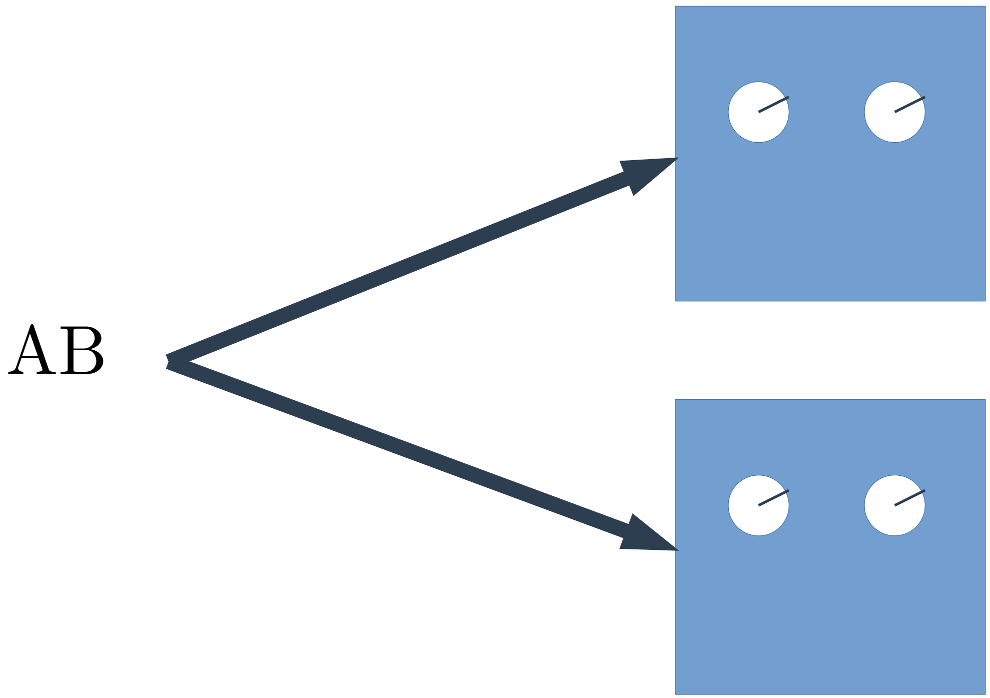
Composite systems

- Systems A and B and AB.
- $\mathbb{C}^{d_\mathsf{A}}, \mathbb{C}^{d_\mathsf{B}}, \mathbb{C}^{d_\mathsf{A} d_\mathsf{B}}$
- $\mathcal{F}_\mathsf{A}, \mathcal{F}_\mathsf{B}, \mathcal{F}_\mathsf{AB}$
- Measurement on system A and B should correspond to a joint measurement on AB.

Joint measurements



Joint measurements



What about composite systems?

- Need a product between systems, and specifically between the OPFs:

$$\star : \mathcal{F}_{\mathsf{A}} \times \mathcal{F}_{\mathsf{B}} \to \mathcal{F}_{\mathsf{AB}}$$

- This product needs to obey certain constraints

$$(f_{\mathsf{A}} \star g_{\mathsf{B}})(\psi_{\mathsf{A}} \otimes \phi_{\mathsf{B}}) = f_{\mathsf{A}}(\psi_{\mathsf{A}}) g_{\mathsf{B}}(\phi_{\mathsf{B}})$$

- Other constraints also follow from operational considerations.

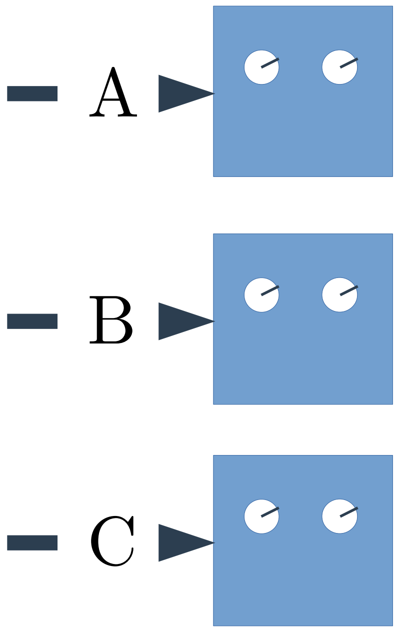
The quantum case:

$$F_{\mathsf{A}}(\psi_{\mathsf{A}}) = \langle \psi_{\mathsf{A}} | \hat{F}_{\mathsf{A}} | \psi_{\mathsf{A}} \rangle$$

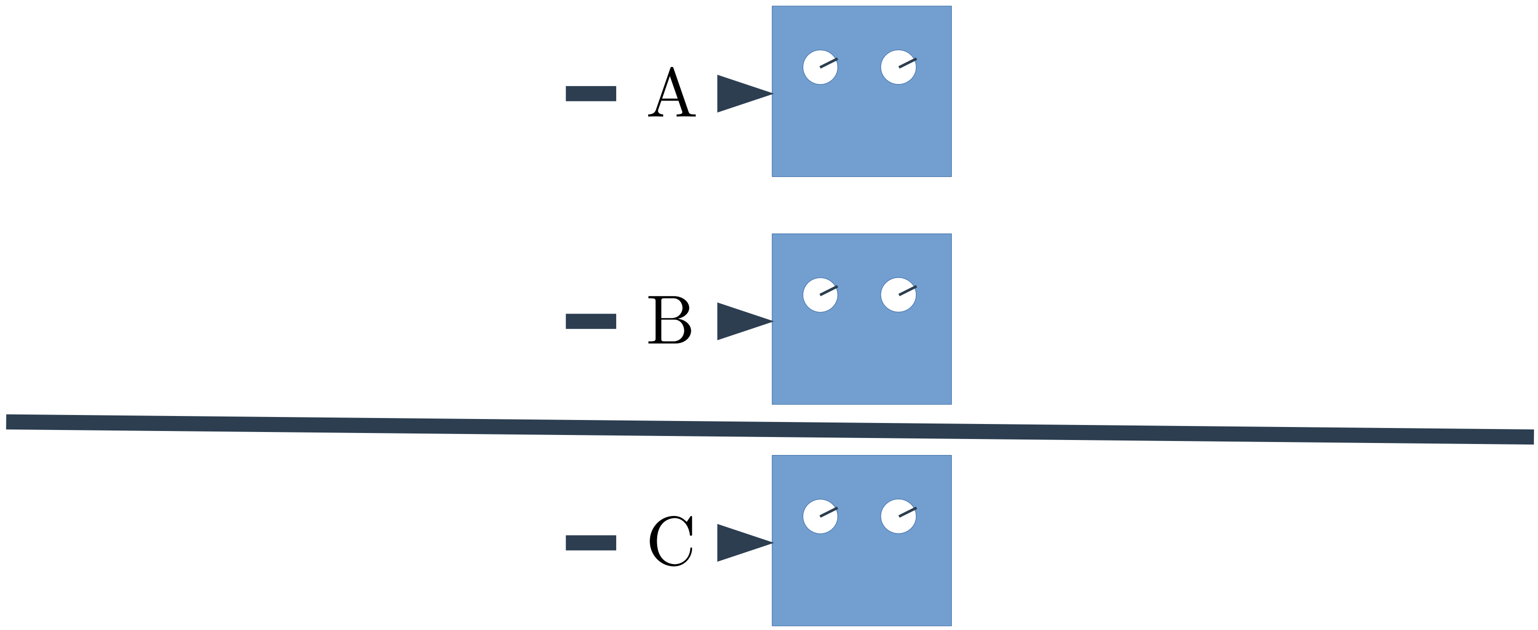
$$G_{\mathsf{B}}(\psi_{\mathsf{B}}) = \langle \psi_{\mathsf{B}} | \hat{G}_{\mathsf{B}} | \psi_{\mathsf{B}} \rangle$$

$$(F_{A} \star G_{\mathsf{B}})(\psi_{\mathsf{AB}}) = \langle \psi_{\mathsf{AB}} | \hat{F}_{\mathsf{A}} \otimes \hat{G}_{\mathsf{B}} | \psi_{\mathsf{AB}} \rangle$$

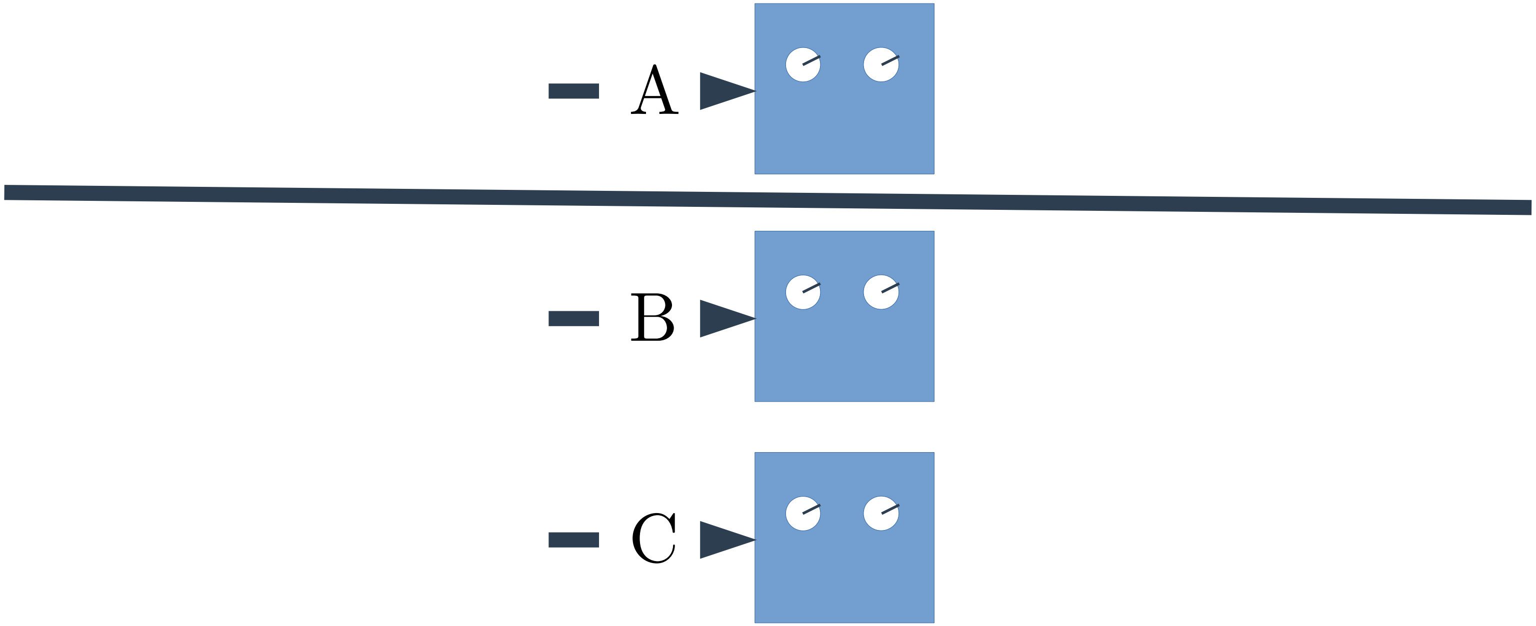
Associativity



Associativity



Associativity



Associativity

- Subjective groupings of devices lead to the same operational predictions
- “pre-operational” constraint

What about composite systems?

$$f_{\mathsf{A}} \star (g_{\mathsf{B}} \star h_{\mathsf{C}}) = (f_{\mathsf{A}} \star g_{\mathsf{B}}) \star h_{\mathsf{C}}$$

The result

- The only OPFs which obey this associativity constraint are the quantum ones.
- Under the assumption that (as a vector space) these sets are finite dimensional.
- Proof makes use of the linear action of the unitary group on $\mathcal{F}_d$

The theorem

The only measurement postulate satisfying the "possibility of state estimation" has OPFs and

$\bigstar$-product of the form

$$F(\psi) = \langle\psi|F|\psi\rangle$$

$$(F_A \star G_{\mathsf{B}})(\psi_{\mathsf{AB}}) = \langle\psi_{\mathsf{AB}}|\hat{F}_{\mathsf{A}} \otimes \hat{G}_{\mathsf{B}}|\psi_{\mathsf{AB}}\rangle$$

for all $\psi \in \mathbb{C}^a$ and $\psi_{AB} \in \mathbb{C}^a \otimes \mathbb{C}^a$ where the $\mathbb{C}^a$ operator F satisfies $0 \leq \hat{F} \leq \mathbb{I}$ and analogously for G.

Conclusion

- The only GPT with the same dynamical structure as quantum theory is quantum theory itself
- The measurement postulates are redundant (assuming the operational framework)

Other approaches

A formal proof of the Born rule from decision-theoretic assumptions

David Wallace

June 15, 2009

Quantum Theory of Probability and Decisions

David Deutsch[1]

Revised February 1999 – to appear in *Proc. R. Soc. Lond.* **A**

Other approaches

PROBABILITIES FROM ENTANGLEMENT, BORN'S RULE $p_k = |\psi_k|^2$ FROM ENVARIANCE

Wojciech Hubert Zurek
Theory Division, MS B210, LANL Los Alamos, NM, 87545, U.S.A.
(Dated: February 1, 2008)

Conclusion

Perspectives on quantum theory

- Quantum logic
- General probabilistic theory
- Many others: Qubism, process theories...
- Also many non-operational approaches

How to study quantum theory

- We have seen how to study GPTs and non-classical logics.
- Quantum theory is just one instance of a GPT or NCL
- What about other (non operational) approaches?
- What is a general branching theory for instance?

Perspectives on quantum theory

- Different approaches provide different insights
- In the case of studying postulates I have used two different approaches: Quantum Logic and GPTs
- We were able to compress the postulates in two directions.
- Is there a more succinct formulation of quantum theory?

Perspectives on quantum theory

- There are multiple mathematical structures in quantum theory
- A lattice structure (of subspaces)
- A convex structure (of state and effect spaces)
- Other structures to be explored: categorical...

Perspectives on quantum theory

- There is a unique logic with the same measurement structure as quantum theory: quantum theory
- There is a unique GPT with the same dynamical structure as quantum theory: quantum theory

Perspectives on quantum theory

- There are redundancies within the postulates of quantum theory.
- But need a background framework within which we can define things and show these redundancies.