Localising Value Indefiniteness with the (Strong) Kochen-Specker Theorem

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joint work with C. S. Calude and K. Svozil

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Outline

Quantum Indeterminism and the Eigenstate–Eigenvalue Link

Kochen-Specker Theorem & Value (In)Definiteness
- What can we conclude from the KS Theorem?

Localising Value Indefiniteness
- Partial VI and Admissibility
- A Stronger KS Theorem

Value Indefiniteness and Contextuality

Quantum Measurements and Indeterminism

Quantum mechanics is *formally* an indeterministic theory

Born rule

Probability of obtaining the outcome 1 when measuring $P_\phi = |\phi\rangle\langle\phi|$ on a state $|\psi\rangle$ is $\Pr(\phi \mid \psi) = \langle\psi| P_\phi |\psi\rangle$.

However, debate about quantum measurement centres on the ontological status of the Born rule

- Often interpreted as an objective probability distribution implying quantum indeterminism
  - Advantage of QRNGs, for example, often attributed to such "intrinsic randomness"

- Born: *"I myself am inclined to give up determinism in the world of atoms."*

Eigenstate–Eigenvalue Link

Eigenvalue–Eigenstate link

A system in a state $|\psi\rangle$ has a definite property of an observable A if and only if $|\psi\rangle$ is an eigenstate of A.

- The "if" direction is relatively uncontentious
- The other direction assumes quantum indeterminism if the Born rule gives probabilities in $(0, 1)$
- We will say an observable is *value definite* if such a property exists (for a given $|\psi\rangle$); otherwise it is *value indefinite*

Why is the E-E link so commonly adopted?

- No-go theorems rule out large classes of classic alternatives

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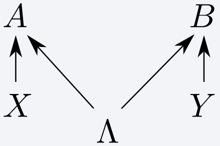
No-Go Theorems: Bell

Local Realism

Any hidden variables must only have local influences:

$$p(x, y|a, b) = \int_\lambda p(a|x, \lambda)p(b|y, \lambda)p(\lambda)d\lambda$$



Bell's Theorem

No local hidden variable model can reproduce the statistical predictions of quantum mechanics.

- Bell's Theorem rules out deterministic local theories, but also indeterministic ones
  - This shows a deeper nonlocal aspect of QM, irrespective of status of determinism

No-Go Theorems: Kochen-Specker

Kochen-Specker Theorem

No deterministic noncontextual hidden variable theory for quantum states in $d \geq 3$ Hilbert space.

- Such theories are *logically* (rather than statistically) inconsistent with QM
  - Can be turned into testable, statistical contradictions
- State independent
- Standard formulation more directly addresses deterministic HVs
  - Tradeoff is dependence on quantum logical structure

I will look more carefully about what can be deduced about (in)determinism from the Kochen-Specker Theorem

The Kochen-Specker Theorem (Informal)

As is common we will restrict ourselves to rank-1 projection observables $P_\psi = |\psi\rangle\langle\psi|$

Context

A *context* in $\mathbb{C}^n$ is a set of n mutually compatible (i.e., commuting) observables.

Kochen-Specker Theorem (Informal)

In dimension $d \geq 3$ Hilbert space there is no hidden variable theory that satisfies the following criteria:

1. Every observable is assigned a definite value of 0 or 1 specifying the measurement outcome;
2. These definite values depend only on the observable in question and not the context that it may be measured in;
3. Exactly one observable in each context is assigned the value 1.

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The Kochen-Specker Theorem (Formal)

We formalise such a HVT with a noncontextual value assignment function $v : \mathcal{O} \to \{0,1\}$, where $\mathcal{O}$ is a set of observables

Kochen-Specker Theorem

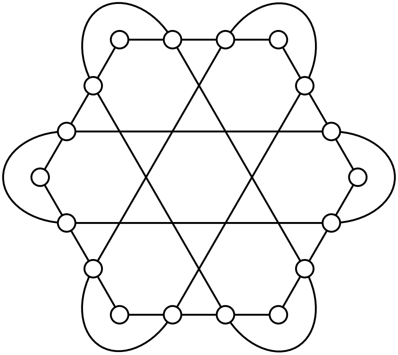
There exists a finite set of observables $\mathcal{O}$ in dimension $d \geq 3$ Hilbert space such that there is no value assignment v satisfying:

1. Value definiteness: v is total, i.e. $v(P)$ is defined for all $P \in \mathcal{O}$;
2. Noncontextuality: v is a function of P only, i.e. $v(P, C_1) = v(P, C_2) = v(P)$ for all contexts $C_1, C_2 \subset \mathcal{O}$ containing P;
3. Quantum correspondence: For every context $C \subset \mathcal{O}$ we have $\sum_{P \in C} v(P) = 1$.

Proving the KS Theorem

Kochen-Specker Theorem (concise)

There exists a finite set $\mathcal{O}$ such that there is no noncontextual value assignment function v on $\mathcal{O}$ satisfying $\sum_{P \in C} v(P) = 1$ for all contexts $C \subset \mathcal{O}$.



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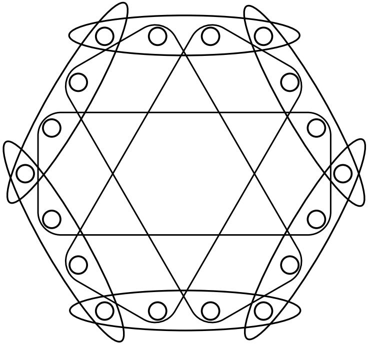

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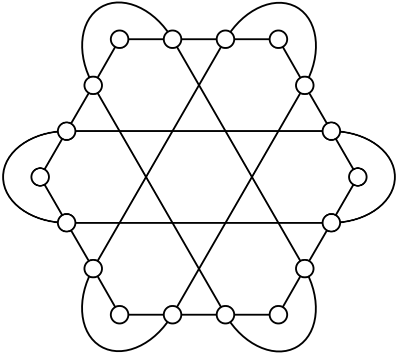

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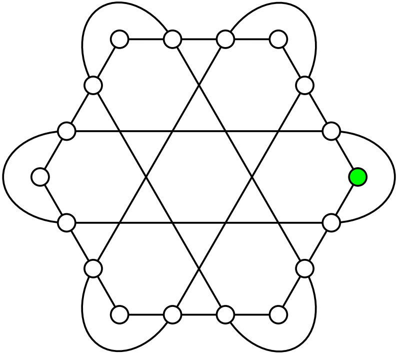

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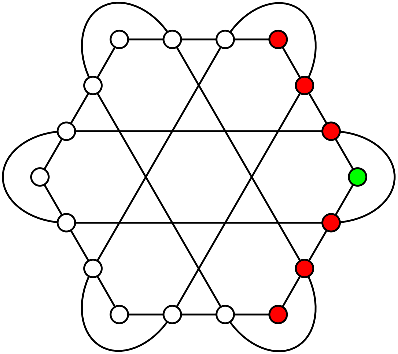

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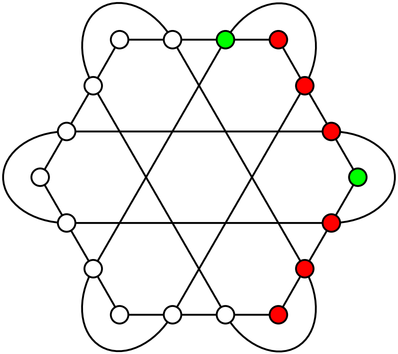

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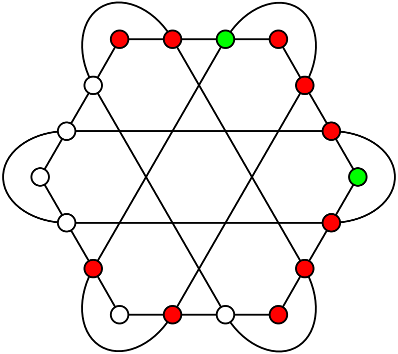

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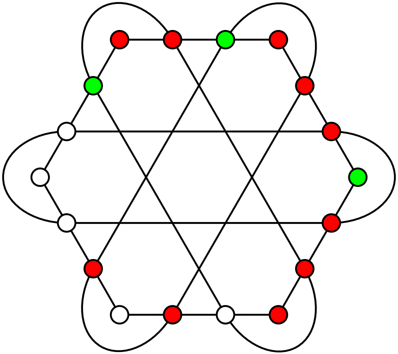

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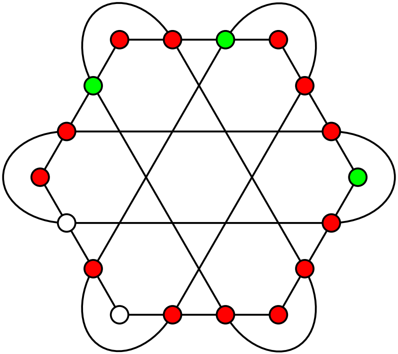

What Does a Contradiction Prove?

What can we conclude from the Kochen-Specker Theorem? Either we reject:

- "Quantum correspondence" ($\sum_{P \in C} v(P) = 1$): but then quantum predictions for compatible measurements are violated
- Noncontextuality: we obtain a contextual hidden variable theories
  - Logically consistent with quantum mechanics and can be physically motivated (e.g. Bohmian mechanics)
- Value definiteness: then *some* observables must be value indefinite and thus the associated measurement indeterministic

This last possibility only shows failure of complete determinism, whereas the E-E link posits that all non-eigenstate measurements are indeterministic

- Can this gap be tightened, or must one simply appeal to symmetry?

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A Digression on Contextuality

"Contextuality" traditionally taken to mean the impossibility of a noncontextual hidden variable theory

- Must VI observables really be contextual, or is indeterminism an escape path?
- Spekkens and co. consider a more general operational approach to (non)contextuality applicable to indeterministic hidden variable theories, mixed states and POVMs to counter this
  - Kochen-Specker Theorem shows impossibility of "outcome deterministic" measurement noncontextuality

Since we are focusing on the implications for determinism, we will not adopt this approach here

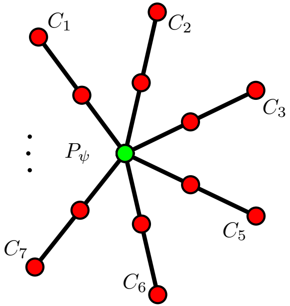
- Could also demand that value assignments reproduce quantum mechanics: $\mathrm{tr}[\rho P] = \sum_\lambda p(\lambda) v_\lambda(P)$

How Much Value Indefiniteness?

Can one show that particular observables must be value indefinite?

- We need to weaken the assumptions we make first
- For any set of observables $\mathcal{O}$ can clearly always find a state $|\psi\rangle$ such that we expect some observables in $\mathcal{O}$ to be value definite

- If system is in state $|\psi\rangle$, we should have $v(P_\psi) = 1$
  - The "safe" direction of the eigenvalue-eigenstate link
- If an observable P_ψ has $v(P_\psi) = 1$, can any incompatible P_ϕ be shown to be value indefinite?



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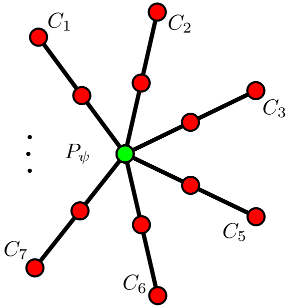

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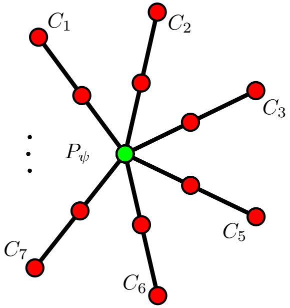

Partial Value Assignment Functions

In focusing on value indefiniteness, we only assume definite values behave noncontextuality: we make no assumption about the behaviour of value indefinite observables

Partial (noncontextual) value assignment functions

A partial value assignment function is a value assignment function $v : \mathcal{O} \to \{0, 1\}$ that is undefined on some observables $P \in \mathcal{O}$.

- $v(P)$ undefined if P is value indefinite
- Value definite observables must be noncontextual as before

Admissibility

How should the condition that for all $C \subset \mathcal{O}$, $\sum_{P \in C} v(P) = 1$ be generalised when contexts may contain value indefinite observables?

- If the value definiteness of an observable P allows the values of other compatible observables to be "predicted with certainty", then those observables should also be value definite

Admissibility of v

A (partial) value assignment function v is admissible if for every context $C \subset \mathcal{O}$:

(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

(b) if there exists a $P \in C$ with $v(P') = 0$ for all $P' \in C \setminus \{P\}$, then $v(P) = 1$.

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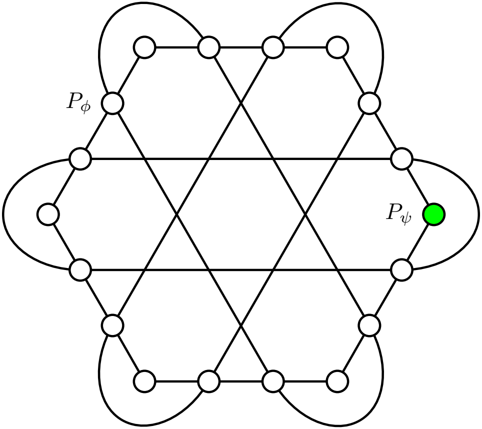
Example

Admissibility provides a weaker way to deduce value definiteness of observables

- Example: Can we have $v(P_\psi) = 1$ and P_ϕ value definite?



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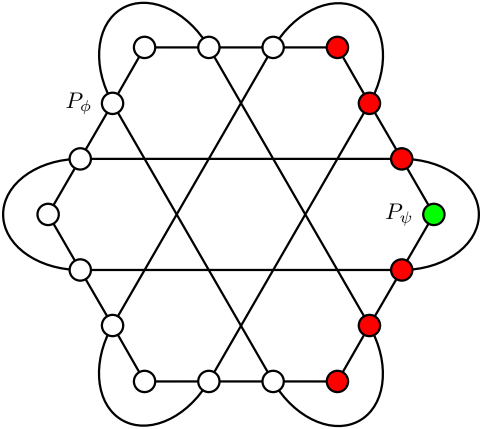

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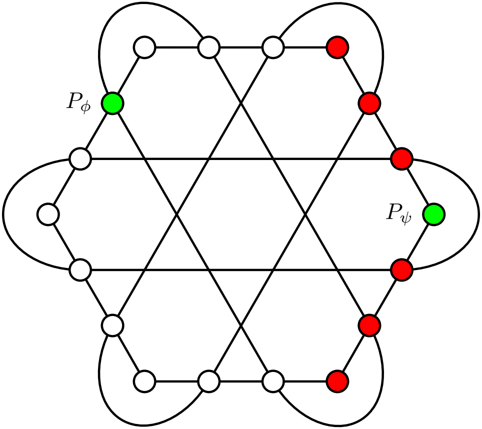

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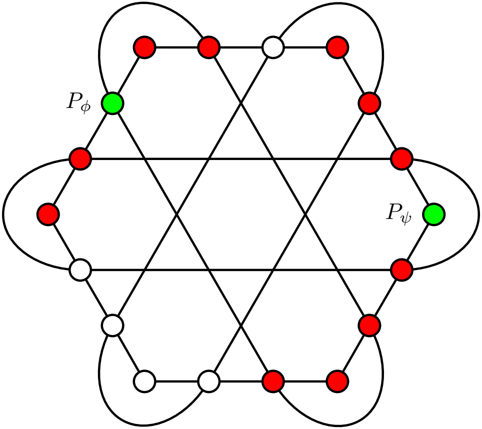

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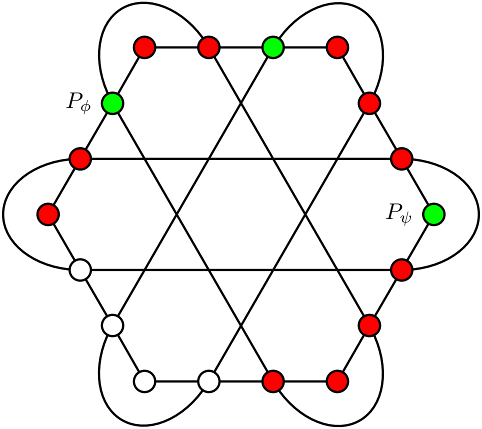

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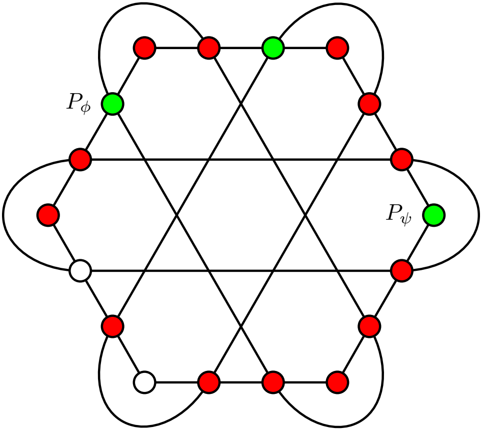

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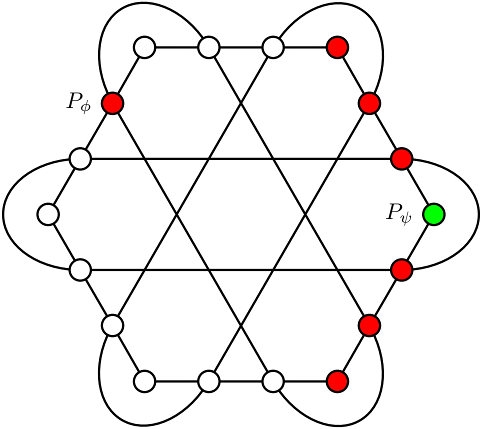

Localising Value Indefiniteness

Strong Kochen-Specker Theorem

Let $n \geq 3$. If an observable P on $\mathbb{C}^n$ is assigned the value 1, then no other incompatible observable can be consistently assigned a definite value at all – i.e., is value indefinite.

If $\mathcal{O}$ is finite then it clearly must depend on both $|\psi\rangle$ and $|\phi\rangle$

We prove in 3 steps:

1. We first prove the explicit case that $|\langle\psi|\phi\rangle| = \frac{1}{\sqrt{2}}$.
2. We prove a reduction for $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$ to the first case.
3. We prove a reduction for the last case of $\frac{1}{\sqrt{2}} < |\langle\psi|\phi\rangle| < 1$.

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Explicit Case

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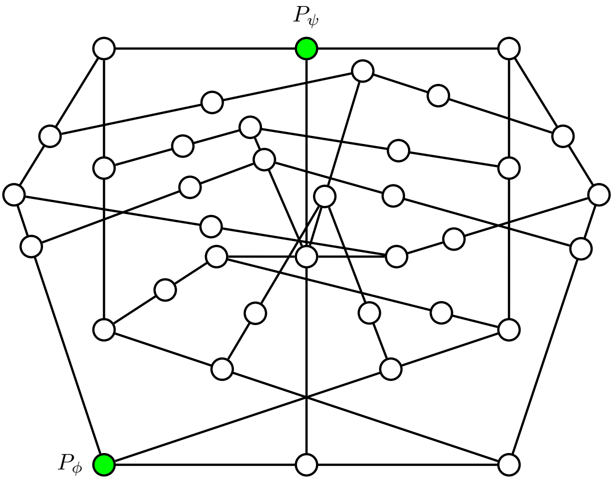
Assuming $v(P_\psi) = v(P_\phi) = 1$, we derive a contradiction:



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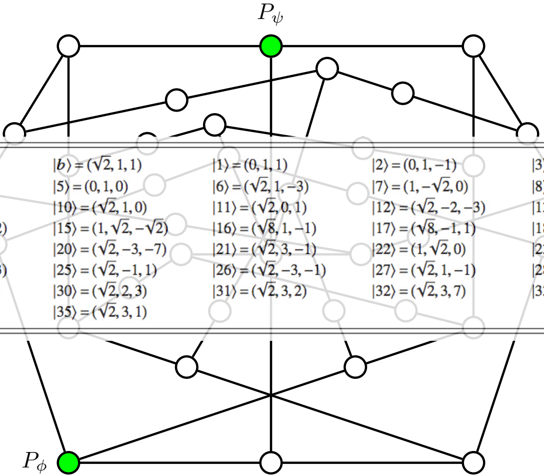

$\|a\rangle = (1,0,0)$	$\|b\rangle = (\sqrt{2},1,1)$	$\|1\rangle = (0,1,1)$	$\|2\rangle = (0,1,-1)$	$\|3\rangle = (\sqrt{2},-1,-1)$
$\|4\rangle = (0,0,1)$	$\|5\rangle = (0,1,0)$	$\|6\rangle = (\sqrt{2},1,-3)$	$\|7\rangle = (1,-\sqrt{2},0)$	$\|8\rangle = (\sqrt{2},-3,1)$
$\|9\rangle = (1,0,-\sqrt{2})$	$\|10\rangle = (\sqrt{2},1,0)$	$\|11\rangle = (\sqrt{2},0,1)$	$\|12\rangle = (\sqrt{2},-2,-3)$	$\|13\rangle = (1,-\sqrt{2},\sqrt{2})$
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$\|29\rangle = (\sqrt{2},0,-1)$	$\|30\rangle = (\sqrt{2},2,3)$	$\|31\rangle = (\sqrt{2},3,2)$	$\|32\rangle = (\sqrt{2},3,7)$	$\|33\rangle = (\sqrt{2},7,3)$
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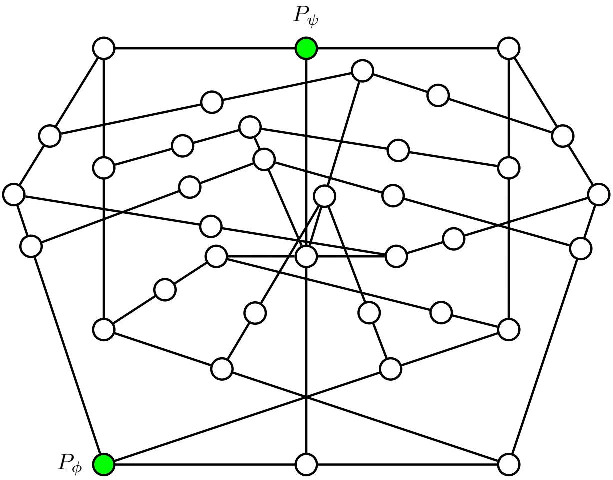

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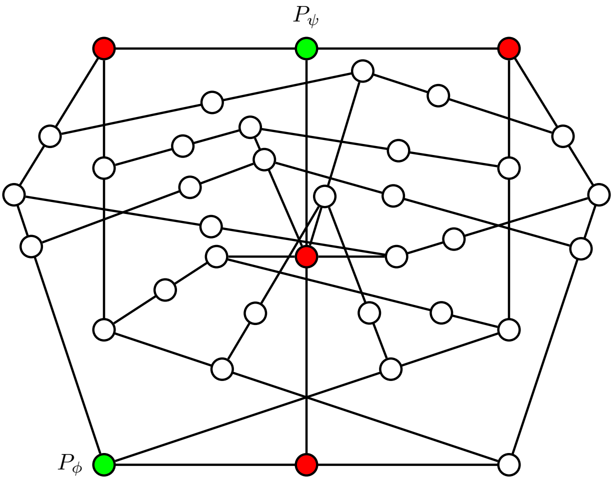

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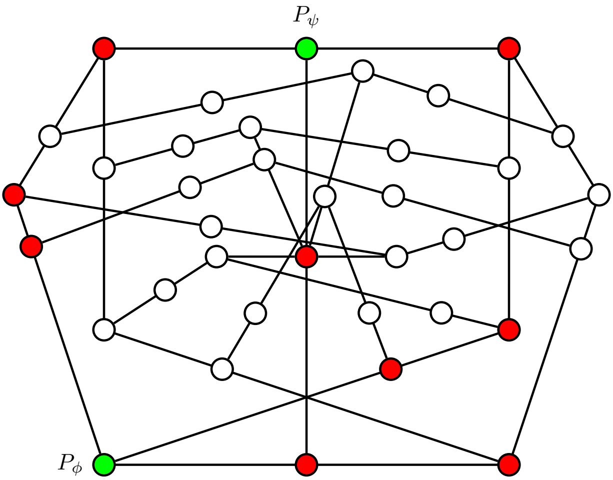

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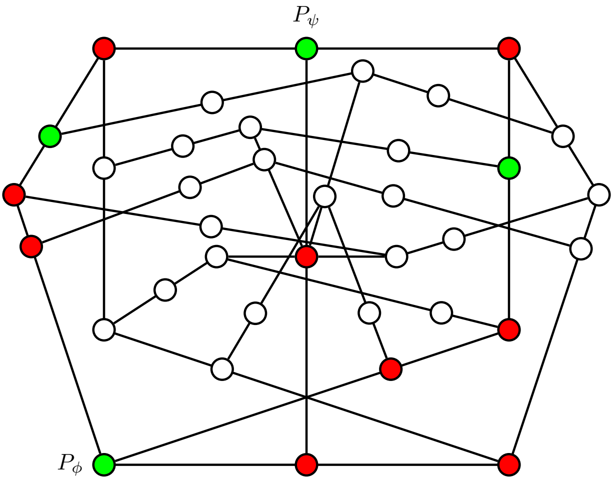

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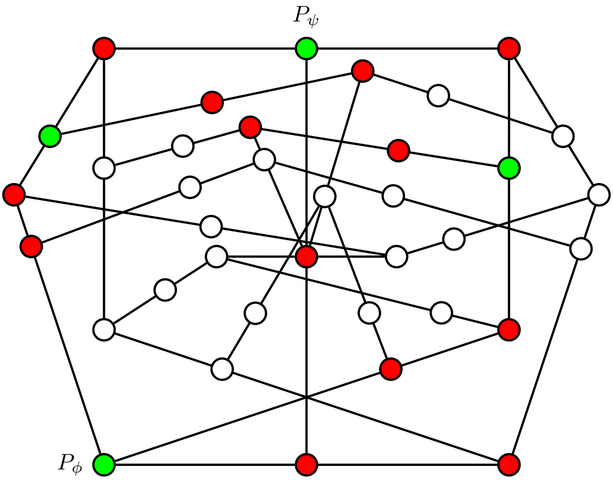

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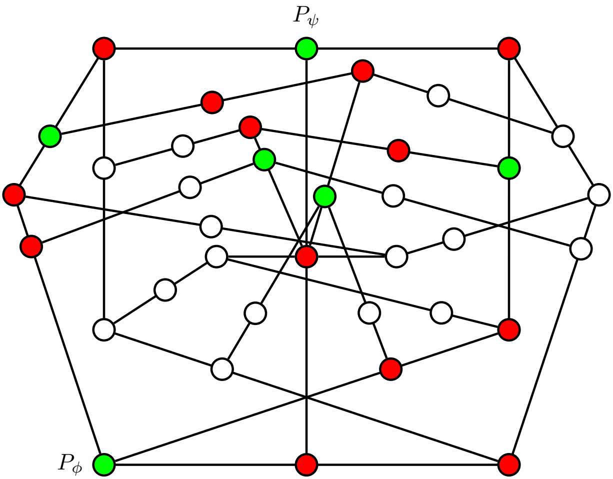

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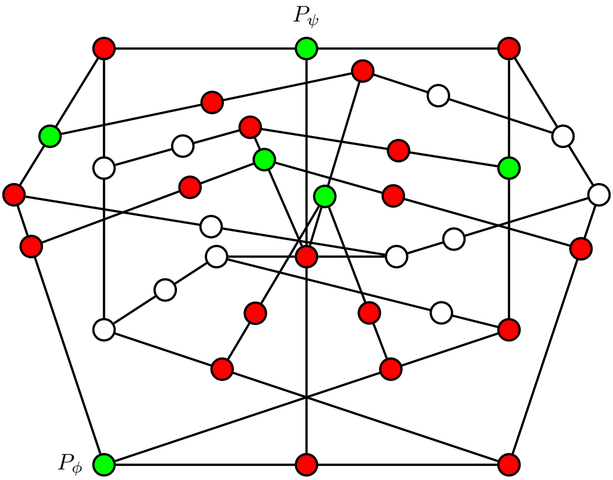

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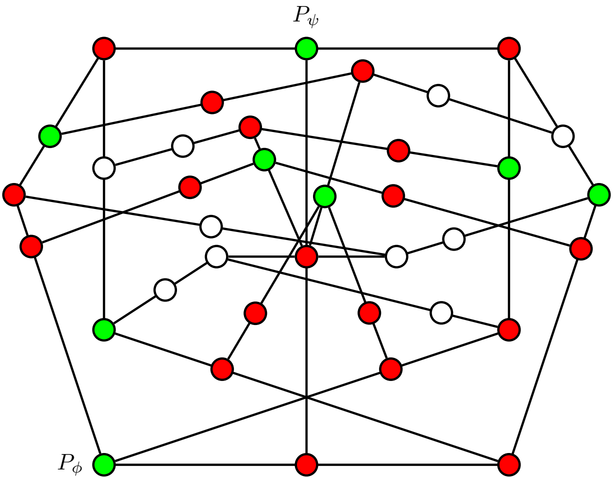

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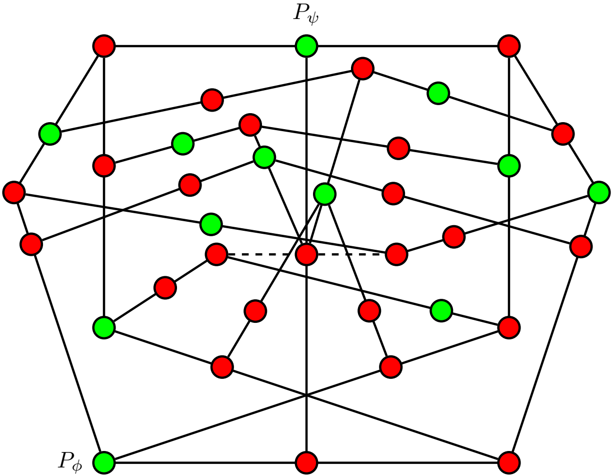

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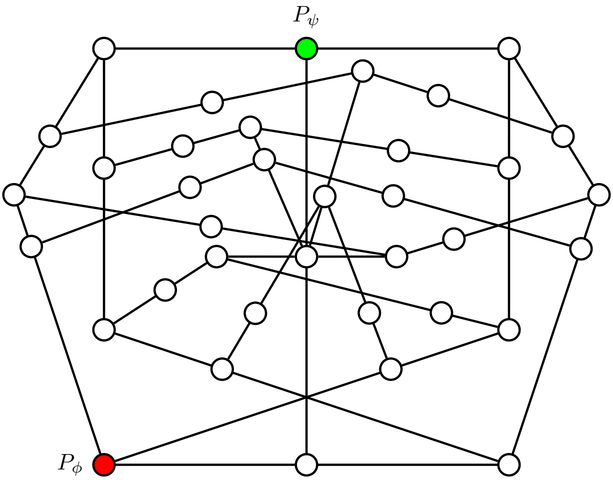

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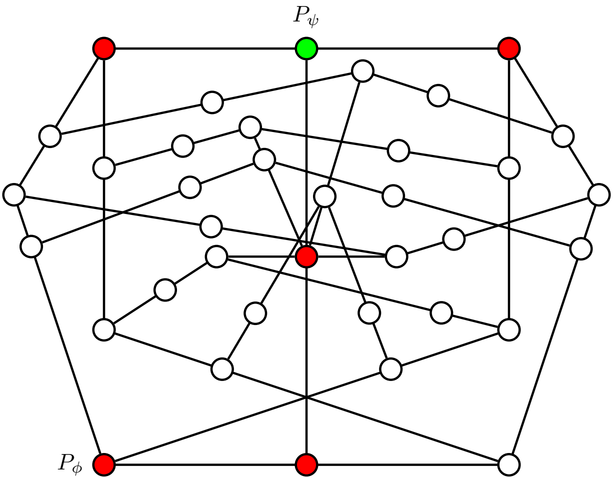

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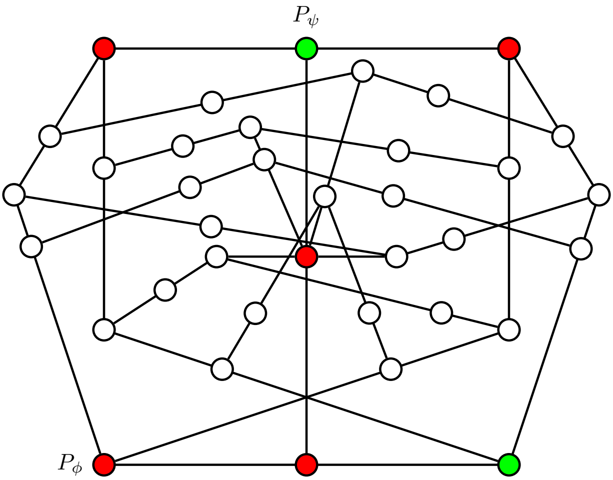

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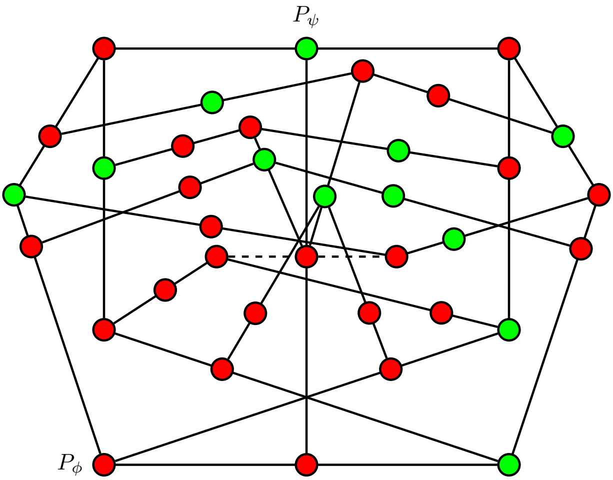

Constructive Reductions

This shows some value indefinite observables can indeed be identified:

Localised Value Indefiniteness

If $|\langle\psi|\phi\rangle| = \frac{1}{\sqrt{2}}$ and $v(P_\psi) = 1$ then P_ϕ must be value indefinite under any admissible value assignment function.

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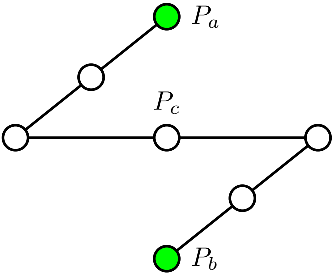

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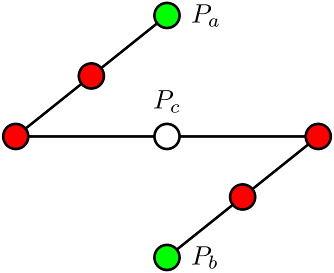

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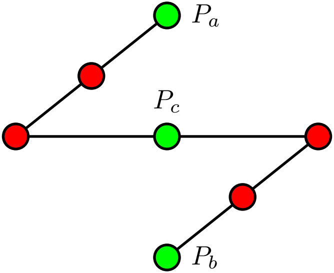
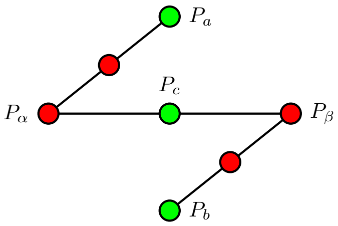


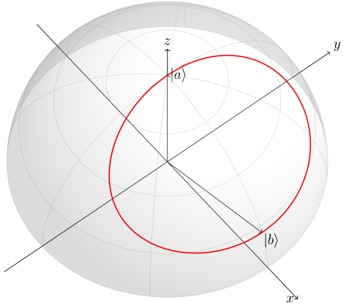
First Reduction: Contraction

- Let $p = \langle a|b\rangle$, $|a\rangle = (0,0,1)$ and $|b\rangle = (\sqrt{1-p^2}, 0, p)$
- Requiring $\langle a|\alpha\rangle = \langle b|\beta\rangle = \langle \alpha|\beta\rangle = \langle \alpha|c\rangle = \langle \beta|c\rangle = 0$ one finds

$$|c_\pm\rangle = (x, \pm\sqrt{1-x^2-z^2}, z) \text{ with } x = \frac{p(1-z^2)}{z\sqrt{1-p^2}}$$

- $|c_\pm\rangle$ is closer to both $|a\rangle$ and $|b\rangle$: $|\langle a|b\rangle| < |\langle a|c_\pm\rangle| < 1$ and $|\langle a|b\rangle| < |\langle b|c_\pm\rangle| < 1$





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Corollary

If $v(P_\psi) = v(P_\phi) = 1$ and $0 < |\langle\psi|\phi\rangle| < \frac{1}{\sqrt{2}}$, then we can find a $|\xi\rangle$ with $\langle\psi|\xi\rangle = \frac{1}{\sqrt{2}}$ and $v(P_\xi) = 1$ under any admissible v.

- The orthogonality relation used in the Contraction Lemma is only nontrivial "building block" for diagrams "forcing" value definiteness
- Forcing the value definiteness of a $|\phi\rangle$ with $|\langle\psi|\phi\rangle|$ close to 1 more difficult
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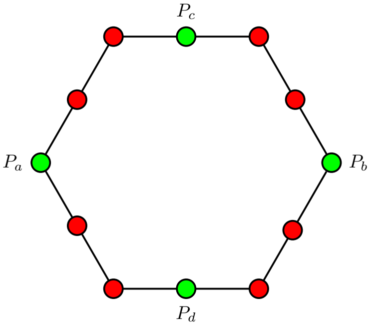
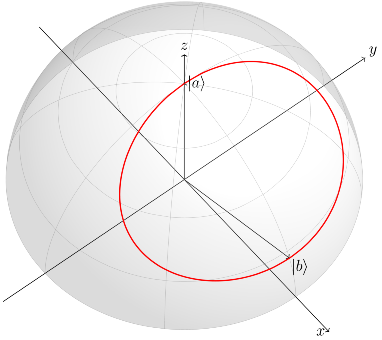
Second Reduction: Expansion

Expansion Lemma

Given $|a\rangle, |b\rangle \in \mathbb{C}^3$ with $\frac{1}{3} < |\langle a|b\rangle| < 1$ there exists $|c\rangle, |d\rangle$ with $0 < |\langle c|d\rangle| < |\langle a|b\rangle|$ and a set of observables $\mathcal{O} \supset \{P_a, P_b, P_c, P_d\}$ such that any admissible v with $v(P_a) = v(P_b) = 1$ must have $v(P_c) = v(P_d) = 1$.





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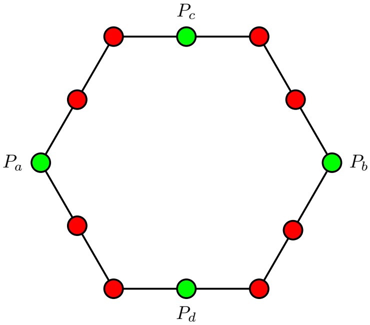

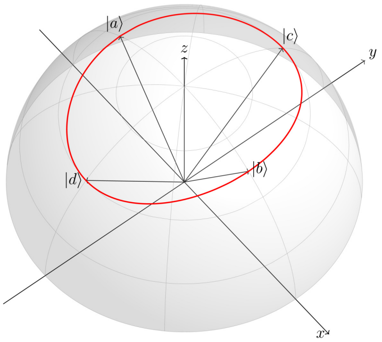

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Taking $|c\rangle$ and $|d\rangle$ as the $|c_\pm\rangle$ vectors from the previous lemma with z chosen so that $|\langle a|c_\pm\rangle| = |\langle b|c_\pm\rangle|$ one can verify that

$$|\langle c|d\rangle| = 3 - \frac{4}{|\langle a|b\rangle| + 1}$$

which for $|\langle a|b\rangle| > \frac{1}{3}$ is positive and $|\langle c|d\rangle| < |\langle a|b\rangle|$ as desired.

- For $|\langle a|b\rangle|$ close to 1 the difference $|\langle a|b\rangle| - |\langle c|d\rangle|$ is small: one must iterate the procedure the procedure to obtain a large enough angle between $|c\rangle$ and $|d\rangle$

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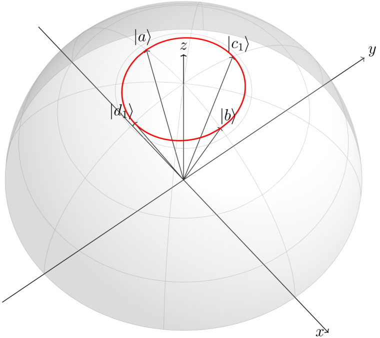

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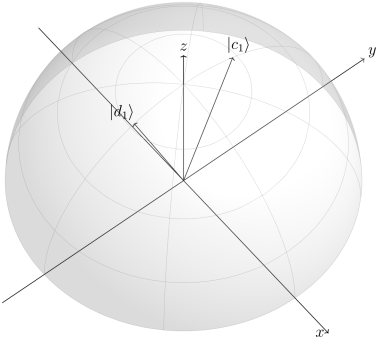

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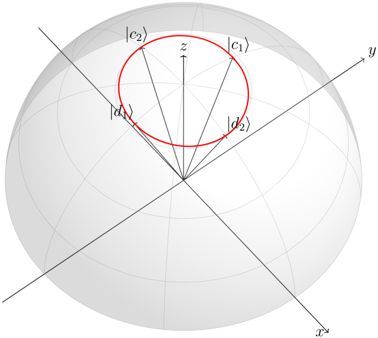

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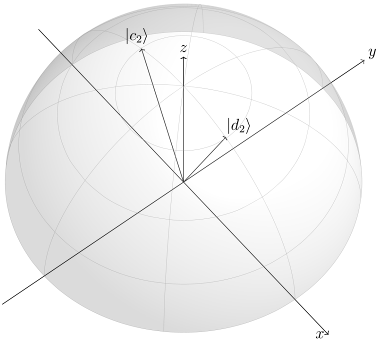

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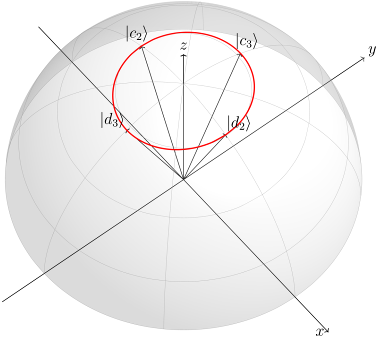

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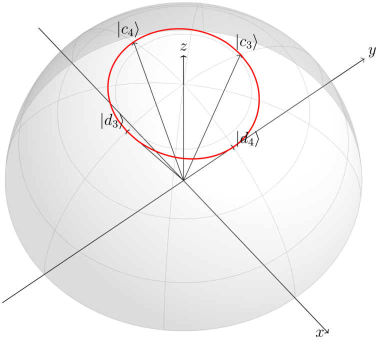

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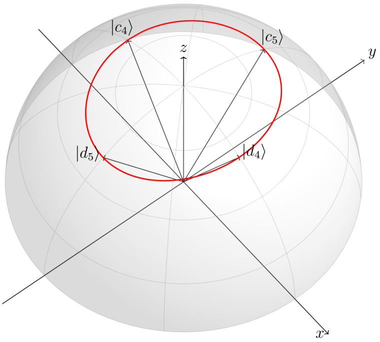

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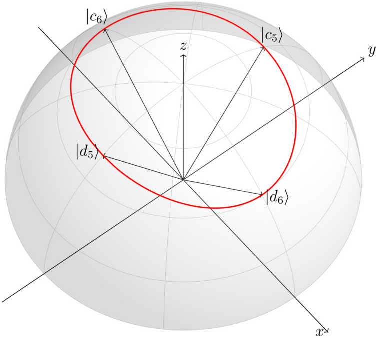

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Non-convergence to a value greater than $1/3$ proved by noting

$$D(u) := u - \left(3 - \frac{4}{u+1}\right)$$

decreasing on $u \in (\frac{1}{3}, 1)$.

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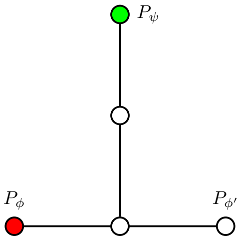

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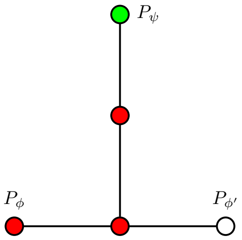

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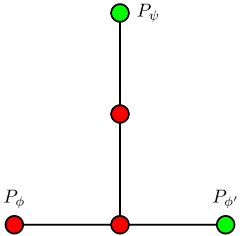

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Interpreting the Result

Eigenstate Value Definiteness

If a system in in state $|\psi\rangle$, then $v(P_\psi) = 1$ for any *faithful* value assignment function.

Interpretation

If a system is in a state $|\psi\rangle$, then the result of measuring an observable A is indeterministic unless $|\psi\rangle$ is an eigenstate of A.

- We assume only that Einstein-type "elements of physical reality" must be noncontextual
  - Contextual hidden variables can't be ruled out
- We assume one direction of the E-E link and, under this assumption, prove the other
- Quantum states must be maximally value indefinite

Experimental Testability

Noncontextuality inequalities have been crucial in making contextuality operationally accessible

$$\langle A_1A_2\rangle + \langle A_2A_3\rangle + \langle A_3A_4\rangle + \langle A_4A_5\rangle + \langle A_5A_1\rangle \geq -3$$

- Derived assuming every observable is noncontextually value definite
  - Quantum correspondence not enforced
  - Admissibility crucial to our argument
- State independent inequalities violated by any state
  - Ensuring one observable in inequality corresponds to prepared state is operationally infeasible
  - Operationally can only use mixed states and POVMs
    - Assigning definite values to POVMs problematic, and can't appeal to eigenstate value definiteness

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Operational Notions of Contextuality

Spekkens showed how to generalise noncontextuality to operational theories

- Operationally equivalent measurements and preparations should be ontologically equivalent
- One motivation: avoid concluding indeterminism instead of contextuality
  - We are precisely interested in indeterminism

Can we draw conclusions about (non)contextuality of value indefinite observables?

$$v_\lambda(P,C) \in \{0,1,\text{undefined}\} \to \xi(k|P_C,\lambda) \in [0,1]$$

- $v_\lambda(P,C) \in \{0,1\} \implies \xi(v_\lambda(P,C)|P_C,\lambda) = 1$
- Correspondence with QM: $\langle\psi|P|\psi\rangle = \sum_\lambda \xi(1|P_C,\lambda)\mu(\lambda|\psi)$

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What about contextuality then?

For a given state $|\psi\rangle$ can always assign all value indefinite P the noncontextual probability

$$\xi(1|P,\lambda) = \langle\psi|P|\psi\rangle$$

so measurement noncontextuality always possible[1]

- We show no P incompatible with P_ψ can have $\xi(1|P,\lambda) \in \{0,1\}$ assuming measurement NC
  - Implied by Gleason's theorem also

Preparation noncontextuality $\implies$ outcome determinism $\implies$ contradiction with measurement noncontextuality

- Do any weaker assumptions on operational noncontextuality imply our assumptions?

[1] R. Spekkens, PRA **71**, 052108 (2005).

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Summary

The Strong Kochen-Specker Theorem allows value indefinite observables to be "localised"

- Shows the *extent* of indeterminism (while KS shows its existence)
- Important for understanding quantum randomness, where individual measurements should be indeterministic
- Doesn't constitute randomness in-and-of itself: quantum randomness is unpredictability that can be seen as arising from this indeterminism

Helps clarify status of quantum indeterminism in QM by bridging the gap between Bell/KS Theorems and the E-E link

- Significance for operational approaches to contextuality remains to be clarified

References

- A. A. Abbott, C. S. Calude & K. Svozil. *A variant of the Kochen-Specker theorem localising value indefiniteness*, J. Math. Phys. 56, 102201 (2015).
- S. Kochen & E. Specker. *The problem of hidden variables in quantum mechanics*, J. Math. Mech. 17, 59 (1967).
- A. Cabello et al. *Bell-Kochen-Specker Theorem: A proof with 18 vectors*, Phys. Lett. A 212, 183 (1996).
- R. Spekkens. *Contextuality for preparations, transformations, and unsharp measurements*, Phys. Rev. A 71, 052108 (2005).
- R. Kunjwal & R. Spekkens. *From the Kochen-Specker theorem to noncontextuality inequalities without assuming determinism*, Phys. Rev. Lett. 115, 110403 (2015).