Localising Value Indefiniteness with the (Strong) Kochen-Specker Theorem

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Zurich, 23 June 2017
Quantum Indeterminism and the Eigenstate–Eigenvalue Link

Kochen-Specker Theorem & Value (In)Definiteness
  What can we conclude from the KS Theorem?

Localising Value Indefiniteness
  Partial VI and Admissibility
  A Stronger KS Theorem

Value Indefiniteness and Contextuality
Quantum mechanics is formally an indeterministic theory

**Born rule**

Probability of obtaining the outcome 1 when measuring $P_{\phi} = |\phi\rangle\langle\phi|$ on a state $|\psi\rangle$ is $\Pr(\phi \mid \psi) = \langle\psi | P_{\phi} |\psi\rangle$.

However, debate about quantum measurement centres on the ontological status of the Born rule

- Often interpreted as an objective probability distribution implying quantum indeterminism
  - Advantage of QRNGs, for example, often attributed to such “intrinsic randomness”

- Born: “I myself am inclined to give up determinism in the world of atoms.”
A system in a state $|\psi\rangle$ has a definite property of an observable $A$ if and only if $|\psi\rangle$ is an eigenstate of $A$.

- The “if” direction is relatively uncontentious.
- The other direction assumes quantum indeterminism if the Born rule gives probabilities in $(0, 1)$.
- We will say an observable is \textit{value definite} if such a property exists (for a given $|\psi\rangle$); otherwise it is \textit{value indefinite}.

Why is the E-E link so commonly adopted?

- No-go theorems rule out large classes of classic alternatives.
Eigenvalue–Eigenstate Link

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No-Go Theorems: Bell

Local Realism

Any hidden variables must only have local influences:

\[ p(x, y|a, b) = \int_{\lambda} p(a|x, \lambda)p(b|y, \lambda)p(\lambda)d\lambda \]

Bell’s Theorem

No local hidden variable model can reproduce the statistical predictions of quantum mechanics.

- Bell’s Theorem rules out deterministic local theories, but also indeterministic ones
  - This shows a deeper nonlocal aspect of QM, irrespective of status of determinism
Kochen-Specker Theorem

No deterministic noncontextual hidden variable theory for quantum states in $d \geq 3$ Hilbert space.

- Such theories are *logically* (rather than statistically) inconsistent with QM
  - Can be turned into testable, statistical contradictions
- State independent
- Standard formulation more directly addresses deterministic HVs
  - Tradeoff is dependence on quantum logical structure

I will look more carefully about what can be deduced about (in)determinism from the Kochen-Specker Theorem
As is common we will restrict ourselves to rank-1 projection observables $P_\psi = |\psi\rangle\langle\psi|$.

A context in $\mathbb{C}^n$ is a set of $n$ mutually compatible (i.e., commuting) observables.

1. Every observable is assigned a definite value of 0 or 1 specifying the measurement outcome;
2. These definite values depend only on the observable in question and not the context that it may be measured in;
3. Exactly one observable in each context is assigned the value 1.
The Kochen-Specker Theorem (Informal)

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**Context**

A *context* in \( \mathbb{C}^n \) is a set of \( n \) mutually compatible (i.e., commuting) observables.

**Kochen-Specker Theorem (Informal)**

In dimension \( d \geq 3 \) Hilbert space there is no hidden variable theory that satisfies the following criteria:

1. Every observable is assigned a definite value of 0 or 1 specifying the measurement outcome;
2. These definite values depend only on the observable in question and not the context that it may be measured in;
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The Kochen-Specker Theorem (Formal)

We formalise such a HVT with a noncontextual value assignment function \( v : \mathcal{O} \rightarrow \{0, 1\} \), where \( \mathcal{O} \) is a set of observables.

Kochen-Specker Theorem

There exists a finite set of observables \( \mathcal{O} \) in dimension \( d \geq 3 \) Hilbert space such that there is no value assignment \( v \) satisfying:

1. Value definiteness: \( v \) is total, i.e. \( v(P) \) is defined for all \( P \in \mathcal{O} \);
2. Noncontextuality: \( v \) is a function of \( P \) only, i.e.
   \[ v(P, C_1) = v(P, C_2) = v(P) \]
   for all contexts \( C_1, C_2 \subset \mathcal{O} \) containing \( P \);
3. Quantum correspondence: For every context \( C \subset \mathcal{O} \) we have
   \[ \sum_{P \in C} v(P) = 1. \]
Kochen-Specker Theorem (concise)

There exists a finite set $\mathcal{O}$ such that there is no noncontextual value assignment function $v$ on $\mathcal{O}$ satisfying $\sum_{P \in C} v(P) = 1$ for all contexts $C \subset \mathcal{O}$.
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![Graph demonstrating the Kochen-Specker theorem](image)
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What can we conclude from the Kochen-Specker Theorem? Either we reject:

- “Quantum correspondence” ($\sum_{P \in C} v(P) = 1$): but then quantum predictions for compatible measurements are violated.
- Noncontextuality: we obtain a contextual hidden variable theories
  - Logically consistent with quantum mechanics and can be physically motivated (e.g. Bohmian mechanics).
- Value definiteness: then some observables must be value indefinite and thus the associated measurement indeterministic.

This last possibility only shows failure of complete determinism, whereas the E-E link posits that all non-eigenstate measurements are indeterministic.

- Can this gap be tightened, or must one simply appeal to symmetry?
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- Can this gap be tightened, or must one simply appeal to symmetry?
“Contextuality” traditionally taken to mean the impossibility of a noncontextual hidden variable theory

- Must VI observables really be contextual, or is indeterminism an escape path?
- Spekkens and co. consider a more general operational approach to (non)contextuality applicable to indeterministic hidden variable theories, mixed states and POVMs to counter this
  - Kochen-Specker Theorem shows impossibility of “outcome deterministic” measurement noncontextuality

Since we are focusing on the implications for determinism, we will not adopt this approach here

- Could also demand that value assignments reproduce quantum mechanics: \( \text{tr}[\rho P] = \sum_\lambda p(\lambda) v_\lambda(P) \)
Can one show that particular observables must be value indefinite?

- We need to weaken the assumptions we make first

- For any set of observables \( \mathcal{O} \) can clearly always find a state \( |\psi\rangle \) such that we expect some observables in \( \mathcal{O} \) to be value definite.

- If system is in state \( |\psi\rangle \), we should have \( v(P_\psi) = 1 \)
  - The “safe” direction of the eigenvalue-eigenstate link

- If an observable \( P_\psi \) has \( v(P_\psi) = 1 \), can any incompatible \( P_\phi \) be shown to be value indefinite?
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  - The “safe” direction of the eigenvalue-eigenstate link.

- If an observable $P_{\psi}$ has $v(P_{\psi}) = 1$, can any incompatible $P_{\phi}$ be shown to be value indefinite?
In focusing on value indefiniteness, we only assume definite values behave noncontextually: we make no assumption about the behaviour of value indefinite observables.

Partial (noncontextual) value assignment functions

A partial value assignment function is a value assignment function $v : \mathcal{O} \rightarrow \{0, 1\}$ that is undefined on some observables $P \in \mathcal{O}$.

- $v(P)$ undefined if $P$ is value indefinite
- Value definite observables must be noncontextual as before
Admissibility

How should the condition that for all $C \subset \mathcal{O}$, $\sum_{P \in C} v(P) = 1$ be generalised when contexts may contain value indefinite observables?

- If the value definiteness of an observable $P$ allows the values of other compatible observables to be “predicted with certainty”, then those observables should also be value definite.

Admissibility of $v$

A (partial) value assignment function $v$ is admissible if for every context $C \subset \mathcal{O}$:

(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

(b) if there exists a $P \in C$ with $v(P') = 0$ for all $P' \in C \setminus \{P\}$, then $v(P) = 1$. 

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Example: Can we have $v(P_\psi) = 1$ and $P_\phi$ value definite?
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Strong Kochen-Specker Theorem

Let \( n \geq 3 \). If an observable \( P \) on \( \mathbb{C}^n \) is assigned the value 1, then no other incompatible observable can be consistently assigned a definite value at all – i.e., is value indefinite.

If \( \mathcal{O} \) is finite then it clearly must depend on both \( |\psi\rangle \) and \( |\phi\rangle \).

We prove in 3 steps:

1. We first prove the explicit case that \( |\langle \psi | \phi \rangle| = \frac{1}{\sqrt{2}} \).
2. We prove a reduction for \( 0 < |\langle \psi | \phi \rangle| < \frac{1}{\sqrt{2}} \) to the first case.
3. We prove a reduction for the last case of \( \frac{1}{\sqrt{2}} < |\langle \psi | \phi \rangle| < 1 \).
Localising Value Indefiniteness

Strong Kochen-Specker Theorem

Let $n \geq 3$ and $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ be states such that $0 < |\langle \psi | \phi \rangle| < 1$. Then there is a finite set of observables $O$ containing $P_\psi$ and $P_\phi$ for which there is no admissible value assignment function on $O$ such that $v(P_\psi) = 1$ and $P_\phi$ is value definite.

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Assuming $v(P_\psi) = v(P_\phi) = 1$, we derive a contradiction:
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This shows some value indefinite observables can indeed be identified:

**Localised Value Indefiniteness**

If $|\langle \psi | \phi \rangle| = \frac{1}{\sqrt{2}}$ and $v(P_\psi) = 1$ then $P_\phi$ must be value indefinite under any admissible value assignment function.

Are there finite Greechie diagrams that allow any $P_\phi$ with $0 < |\langle \psi | \phi \rangle| < 1$ to be shown to be value indefinite?

- In general, Greechie diagrams realisable for a many sets of observables
- We were unable to find any simple such example
- We thus need reductions:
  - For any $|\phi\rangle$, construct a set of observables containing $P_\psi$, $P_\phi$ and $P_\xi$ with $|\langle \psi | \xi \rangle| = \frac{1}{\sqrt{2}}$ such that if $v(P_\psi) = 1$ and $P_\phi$ value definite, then $P_\xi$ must be value definite too
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Constructive Reductions

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First Reduction: Contraction

Contraction Lemma

Given $|a\rangle, |b\rangle \in \mathbb{C}^3$ and $z \in \mathbb{C}$ with $0 < |\langle a|b \rangle| < |z| < 1$ there exists a $|c\rangle$ with $\langle a|c \rangle = z$ and a set of observables $\mathcal{O} \supset \{P_a, P_b, P_c\}$ such that any admissible value assignment function with $v(P_a) = v(P_b) = 1$ must have $v(P_c) = 1$ also.

The simplest Greechie diagram with this property is:
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![Greechie Diagram](image-url)
First Reduction: Contraction

- Let \( p = \langle a|b \rangle, \langle a \rangle = (0, 0, 1) \) and \( |b\rangle = (\sqrt{1 - p^2}, 0, p) \)
- Requiring \( \langle a|\alpha \rangle = \langle b|\beta \rangle = \langle \alpha|\beta \rangle = \langle \alpha|c \rangle = \langle \beta|c \rangle = 0 \) one finds \( |c_{\pm}\rangle = (x, \pm \sqrt{1 - x^2 - z^2}, z) \) with \( x = \frac{p(1 - z^2)}{z\sqrt{1 - p^2}} \)
- \( |c_{\pm}\rangle \) is closer to both \( |a\rangle \) and \( |b\rangle \): \( |\langle a|b \rangle| < |\langle a|c_{\pm}\rangle| < 1 \) and \( |\langle a|b \rangle| < |\langle b|c_{\pm}\rangle| < 1 \)
First Reduction: Contraction

Corollary

If \( v(P_\psi) = v(P_\phi) = 1 \) and \( 0 < |\langle \psi | \phi \rangle| < \frac{1}{\sqrt{2}} \), then we can find a \( |\xi\rangle \) with \( \langle \psi | \xi \rangle = \frac{1}{\sqrt{2}} \) and \( v(P_\xi) = 1 \) under any admissible \( v \).

- The orthogonality relation used in the Contraction Lemma is only nontrivial “building block” for diagrams “forcing” value definiteness
- Forcing the value definiteness of a \( |\phi\rangle \) with \( |\langle \psi | \phi \rangle| \) close to 1 more difficult
- No single diagram appears sufficient
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Second Reduction: Expansion

Expansion Lemma

Given \(|a\rangle, |b\rangle \in \mathbb{C}^3\) with \(\frac{1}{3} < |\langle a|b\rangle| < 1\) there exists \(|c\rangle, |d\rangle\) with \(0 < |\langle c|d\rangle| < |\langle a|b\rangle|\) and a set of observables \(\mathcal{O} \supset \{P_a, P_b, P_c, P_d\}\) such that any admissible \(v\) with \(v(P_a) = v(P_b) = 1\) must have \(v(P_c) = v(P_d) = 1\).
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Taking $|c\rangle$ and $|d\rangle$ as the $|c_{\pm}\rangle$ vectors from the previous lemma with $z$ chosen so that $|\langle a|c_{\pm}\rangle| = |\langle b|c_{\pm}\rangle|$ one can verify that

$$|\langle c|d\rangle| = 3 - \frac{4}{|\langle a|b\rangle| + 1}$$

which for $|\langle a|b\rangle| > \frac{1}{3}$ is positive and $|\langle c|d\rangle| < |\langle a|b\rangle|$ as desired.

- For $|\langle a|b\rangle|$ close to 1 the difference $|\langle a|b\rangle| - |\langle c|d\rangle|$ is small: one must iterate the procedure to obtain a large enough angle between $|c\rangle$ and $|d\rangle$. 
Iteration Lemma

Given $|a\rangle, |b\rangle \in \mathbb{C}^3$ with $\frac{1}{3} < |\langle a|b\rangle| < 1$ there exists $|c\rangle, |d\rangle$ with $0 < |\langle c|d\rangle| \leq \frac{1}{3}$ and a set of observables $\mathcal{O} \supset \{P_a, P_b, P_c, P_d\}$ such that any admissible $v$ with $v(P_a) = v(P_b) = 1$ must have $v(P_c) = v(P_d) = 1$. 
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Second Reduction: Expansion

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Completing the Proof

Non-convergence to a value greater than $1/3$ proved by noting

\[ D(u) := u - \left( 3 - \frac{4}{u + 1} \right) \]

decreasing on $u \in \left( \frac{1}{3}, 1 \right)$.

\[ \text{Note that } |\langle a| b \rangle| - |\langle c| d \rangle| = D(|\langle a| b \rangle|), \text{ etc.} \]

Corollary

If $v(P_\psi) = v(P_\phi) = 1$ and $0 < \frac{1}{\sqrt{2}} < |\langle \psi| \phi \rangle|$, then we can find a $|\xi \rangle$ with $\langle \psi| \xi \rangle = \frac{1}{\sqrt{2}}$ and $v(P_\xi) = 1$ under any admissible $v$. 
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Completing the Proof

To complete the proof, need to show $\nu(P_\xi) = 0$ also leads to a contradiction.

It is easy to see that $\nu(P_\phi) = 0$ implies $\nu(P_\phi') = 1$ for some $|\phi'\rangle$.

For any given $P_\phi$ incompatible with $P_\psi$, the reductions give constructively a finite set of observables $\mathcal{O}$ containing $P_\psi$ and $P_\phi$ with the desired properties.
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Interpreting the Result

**Eigenstate Value Definiteness**

If a system is in state $|\psi\rangle$, then $\nu(P_\psi) = 1$ for any *faithful* value assignment function.

**Interpretation**

If a system is in a state $|\psi\rangle$, then the result of measuring an observable $A$ is indeterministic unless $|\psi\rangle$ is an eigenstate of $A$.

- We assume only that Einstein-type “elements of physical reality” must be noncontextual
  - Contextual hidden variables can’t be ruled out
- We assume one direction of the E-E link and, under this assumption, prove the other
- Quantum states must be maximally value indefinite
Noncontextuality inequalities have been crucial in making contextuality operationally accessible

\[ \langle A_1 A_2 \rangle + \langle A_2 A_3 \rangle + \langle A_3 A_4 \rangle + \langle A_4 A_5 \rangle + \langle A_5 A_1 \rangle \geq -3 \]

- Derived assuming every observable is noncontextually value definite
  - Quantum correspondence not enforced
  - Admissibility crucial to our argument

- State independent inequalities violated by any state
  - Ensuring one observable in inequality corresponds to prepared state is operationally infeasible
  - Operationally can only use mixed states and POVMs
    - Assigning definite values to POVMs problematic, and can’t appeal to eigenstate value definiteness
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Spekkens showed how to generalise noncontextuality to operational theories

- Operationally equivalent measurements and preparations should be ontologically equivalent
- One motivation: avoid concluding indeterminism instead of contextuality
  - We are precisely interested in indeterminism

Can we draw conclusions about (non)contextuality of value indefinite observables?

\[ v_\lambda(P, C) \in \{0, 1, \text{undefined}\} \rightarrow \xi(k|P_C, \lambda) \in [0, 1] \]

- \[ v_\lambda(P, C) \in \{0, 1\} \implies \xi(v_\lambda(P, C)|P_C, \lambda) = 1 \]
- Correspondence with QM: \[ \langle \psi | P | \psi \rangle = \sum_\lambda \xi(1|P_C, \lambda) \mu(\lambda|\psi) \]
Operational Notions of Contextuality

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For a given state $|\psi\rangle$ can always assign all value indefinite $P$ the noncontextual probability

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so measurement noncontextuality always possible$^1$

- We show no $P$ incompatible with $P_\psi$ can have $\xi(1|P, \lambda) \in \{0, 1\}$ assuming measurement NC
  - Implied by Gleason’s theorem also

Preparation noncontextuality $\implies$ outcome determinism $\implies$
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\(^1\)R. Spekkens, PRA 71, 052108 (2005).
The Strong Kochen-Specker Theorem allows value indefinite observables to be “localised”

- Shows the extent of indeterminism (while KS shows its existence)
- Important for understanding quantum randomness, where individual measurements should be indeterministic
- Doesn’t constitute randomness in-and-of itself: quantum randomness is unpredictability that can be seen as arising from this indeterminism

Helps clarify status of quantum indeterminism in QM by bridging the gap between Bell/KS Theorems and the E-E link

- Significance for operational approaches to contextuality remains to be clarified


