"About your cat, Mr. Schrodinger—I have good news and bad news."

Bohr

“Atomic physics deprives of all meaning such *inherent attributes* as the idealization of classical physics would ascribe to such objects.”

A Reconstruction of Quantum Mechanics, *Foundations of Physics*, (2015), p551.

A.Kolmogorov, “ Foundations of the Theory of Probability”   1933

Probability theory is based on a probability measure, i.e. a function

$$p: B \longrightarrow [0,1]$$

with domain B a Boolean σ-algebra, such that

$p(1) = 1$

$p(\cup a_i) = \Sigma p(a_i)$

for $a_1, a_2, \ldots$ in B such that $a_i \,.\, a_j = 0$. $i \neq j$

In an experiment we measure a σ-algebra of properties of a system and the theory tells us how to compute the probability measure on it

Quantum Mechanics: Born’s Rule. Consider a system in a pure state given by the vector Ω and an observable A with a discrete spectrum

Let $A = \sum \lambda_i P_i$ be the spectral decomposition of A. Then the probability that a measurement of A will give the value λ_k is $|| P_k \Omega ||^2$,

and the system after the measurement is in the state $P_k \Omega$. (Projection Rule)

The *interaction algebra* is the Boolean σ-algebra B generated by

$\{P_1, P_2, P_3 \ldots\}$.

The Born Rule defines a probability measure p_Ω: B⟶ [0, 1] given by

$p_\Omega(VP_i) = \Sigma||P_i \Omega||^2$.

The result of the experiment gives a truth value to the projections, say, P_k is true and all other P_i are false and hence to the interaction algebra B.

Different experiments on the system yield different Boolean σ-algebras.e.g. by varying the direction of the magnetic field in the Stern-Gerlach experiment on the spin 1 particle.

For a classical system, we measure intrinsic properties that the system possesses. Hence any two properties can be measured without disturbance, and thus the union of all the σ-algebras forms a σ-algebra. B(Ω) = σ-algebra of Borel sets of phase space Ω.

In Kochen-Specker 1967 we construct a quantum system with a family of Boolean algebras which cannot be imbedded into a single Boolean algebra.

For spin 1 system S_x^2, S_y^2, S_z^2 commute and can be simultaneously measured. E.g.

Measure observable $S_x^2 - S_y^2$ with spectral decomposition $(+1) S_x^2 + (-1) S_y^2 + (0) S_z^2$

With 8 element interaction algebra generated by S_x^2, S_y^2, S_z^2.

There are 40 such observables whose interaction algebras cannot all be imbedded in a single Boolean σ- algebra.

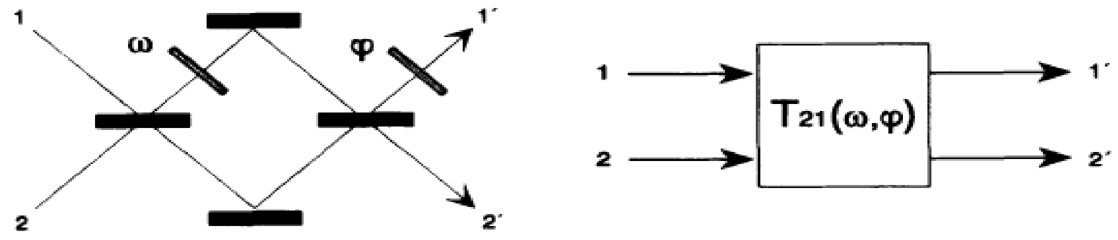
We replace the σ-algebra of classical intrinsic properties by the minimal structure needed to describe all the σ-algebras of measuring observables: the union of these σ-algebras.

Given a family of Boolean σ-algebras, we call their union UB a **Boolean σ-complex.**

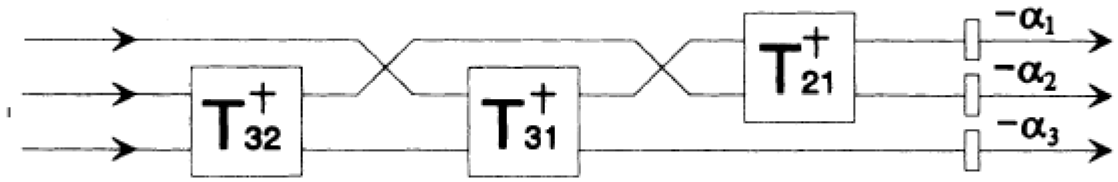
For a Hilbert space $\mathcal{H}$ a set of projections which is closed under countable products and orthogonal complement 1-P forms a Boolean σ-algebra. The family of all such Boolean σ-algebras forms the *Boolean σ- complex Q($\mathcal{H}$).*

<u>Realization of Q($\mathcal{H}$) by Interferometry</u>

Mach-Zender Interferometer with phase shiifters



$$\begin{pmatrix} k_1' \\ k_2' \end{pmatrix} = \begin{pmatrix} e^{\imath\phi}\sin\omega & e^{\imath\phi}\cos\omega \\ \cos\omega & -\sin\omega \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$



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States

Classical Mechanics B(Ω): A (mixed) state is a probability measure on the σ-algebra B(Ω).

B(Ω) is the σ-algebra generated by the open sets of Phase space Ω.

General Theory Q: A *state* of a system with a σ-complex Q is a probability measure on Q, i.e. a map

$$p: Q \longrightarrow [0,1]$$

such that the restriction p|B of p to any σ-algebra B in Q is a probability measure.

Quantum Mechanics For Q = Q($\mathcal{H}$), there is a one-one correspondence between states p on Q($\mathcal{H}$) and density operators (i.e. positive Hermitean operators of trace 1) w on $\mathcal{H}$ such that

$$\mathrm{p(x) = tr(wx), for\ all\ x} \in \mathrm{Q.}$$

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Pure and Mixed States

The set of states is closed under the formation of convex linear combinations, i.e. if p_1, p_2, … are states then so is $\sum c_i p_i$, for for positive c_i, with $\sum c_i = 1$. The extreme points of the convex set of states of a system are those that cannot be written as a non-trivial convex combination of states of the system.

General Theory Q: A *pure state* of a system is an extreme point of the convex set of all states of the system.

If Q = B(Ω), then the pure states have the form

$$p(s) = \begin{cases} 1 \text{ if } \omega \in s \\ 0 \text{ if } \omega \in s' \end{cases}$$

for some $\omega \in \Omega$. Hence,

Classical Mechanics: Assume Q = B(Ω). There is a one-one correspondence between pure states of B(Ω) and points in Ω.

Quantum Mechanics: Assume Q = Q($\mathcal{H}$). The pure states of Q($\mathcal{H}$) are in one-one correspondence with the rank one projection operators or, equivalently, the rays of $\mathcal{H}$.

The state p is given by rank 1 projection P_ψ onto the one-dimensional subspace spanned by the unit vector ψ, so that $P_\psi = |\psi\rangle\langle\psi|$ and

$p(x) = tr(P_\psi x) = \langle\psi, x\psi\rangle$.

State Preparation and Conditional Probability

Classical Mechanics B(Ω): For every state p(x) on B(Ω) and every y∈B(Ω) such that $p(y) \neq 0$, the *state conditioned on y* is the state

$$p(x \mid y) = p(x\,.\,y)/p(y).$$

General Theory Q: Let p(x) be state on a σ-complex Q and y ∈ Q such that $p(y) \neq 0$. By a *state conditioned on y*, we mean a state $p(x \mid y)$ such that for every σ-algebra B in Q containing y and every x ∈ B,

$$p(x \mid y) = p(x\,.\,y)/p(y)$$

Quantum Mechamics: Assume $Q = Q(\mathcal{H})$. For every state p(x) and element y in Q such that $p(y) \neq 0$, there exists a unique state $p(x \mid y)$ conditioned on y.

If w is the density operator corresponding to the state p(x), then ywy/tr(ywy) is the density operator corresponding to state $p(x \mid y)$.

The change of state from density operator w to to density operator ywy/tr(wy) is the reduction called *The Von Neumann-Lüder's Projection Rule.* Here it appears as the extrinsic counterpart of conditioning a probability to a given property.

Classical vs Quantum Conditioning

Let $y = \Sigma y_i$ where $y_i \,.\, y_j = 0$ for $i \neq j$. Then

Classical Mechanics: $p(x|y) = \Sigma p(x|y_i)p(y_i|y)$. This rule is called *The Law of Total Probability.*

Quantum Mechanics:

$p(x|y) = tr(ywyx)/tr(wy)$.

$= tr(\Sigma y_i w \Sigma y_i x) = tr(\Sigma_{i,j} y_i w y_j x) = \Sigma tr(y_i w y_i x) + \Sigma_{i \neq j} tr(y_i w y_j x)$

$= \Sigma p(x|y_i)\, p(y_i|y) + \Sigma_{i \neq j} tr(y_i w y_j x)$.

(interference term)

Example: The Two Slit Experiment

y_i = projection operator P_i of position at slit i, i =1,2.

x = projection operator P_s of position s at detection screen.

Observables

A classical observable is defined as a real-valued function $f\colon \Omega \longrightarrow \mathbb{R}$ on the phase space Ω of the system. To avoid pathological, non-measurable functions, f is assumed to be a Borel function, i.e. a function such that $f^{-1}(s) \in B(\Omega)$, a Borel set in Ω, for every set s in the σ-algebra $B(\mathbb{R})$ of Borel sets generated by the open intervals of $\mathbb{R}$.

The inverse function $f^{-1}\colon B(\mathbb{R}) \longrightarrow B(\Omega)$ is easily seen to preserve the Boolean σ-operations, i.e. to be a homomorphism. Moreover, as we see below, this homomorphism allows us to recover the function f.

Classical Mechanics $B(\Omega)$: An observable is a Borel function $f\colon \Omega \longrightarrow \mathbb{R}$.

There is a one-one correspondence between observables f and homomorphisms $u\colon B(\mathbb{R}) \longrightarrow B(\Omega)$, such that $u = f^{-1}$.

[i.e. maps satisfying $u(s')=u(s)'$, $u(\mathbf{U}s_i) = \mathbf{U}u(s_i)$, for all $s, s_1, s_2, \ldots$ in $B(\mathbb{R})$.]

General Theory Q: An observable u of a system with σ-algebra Q is a homomorphism

$$u\colon B(\mathbb{R}) \longrightarrow Q$$

Quantum Mechanics: Assume $Q=Q(\mathcal{H})$. Then there is a one-one correspondence between observables $u\colon B(\mathbb{R}) \longrightarrow Q(\mathcal{H})$ and Hermitean operators A such that, given u, $A=\int \lambda dP_\lambda$, where $P_\lambda = u((\infty, \lambda])$.

Symmetries

Definition. A *symmetry* of a σ-complex Q is an automorphism of Q, i.e.a one-to-one transformation
$\sigma: Q \longrightarrow Q$

of Q onto Q such that

$$\sigma(a') = \sigma(a)'$$

and for all σ-algebras B in Q and all a_1, a_2, ... in B

$$\sigma(\mathbf{U} a_i) = \mathbf{U}\sigma(a_i).$$

A symmetry σ defines a natural map $p \longrightarrow p_\sigma$ on the states of Q, namely,

$$p_\sigma(x) = p(\sigma(x)), \text{ for all } x \in Q.$$

Quantum Mechanics: Assume $Q = Q(\mathcal{H})$. Then there is a one-one correspondence between symmetries $\sigma: Q(\mathcal{H}) \longrightarrow Q(\mathcal{H})$ and unitary or anti-unitary operators such that $\sigma(x) = uxu^{-1}$, for all $x \in Q(\mathcal{H})$.

If a state p corresponds to the density operator w, then

$$p_\sigma(x) = p(\sigma(x)) = tr(wuxu^{-1}) = tr(u^{-1}wux),$$

so that the state p_σ corresponds to the density operator $u^{-1}wu$.

Dynamics.

Now that we have shown that the symmetries of Q($\mathcal{H}$) are implemented by symmetries of $\mathcal{H}$, we may time symmetry to introduce a dynamics for systems.

To define dynamical evolution, we consider systems that are invariant under time translation. For such systems, there is no absolute time, only time differences. The difference from time 0 to time t is given by a symmetry σ_t: Q →Q, since the structure of the system of properties is indistinguishable at two values of time.

The passage of time is given by a continuous representation of the additive group ℝ of real numbers into the group Aut(Q) of automorphisms of Q under composition:

$$\mathbb{R} \longrightarrow \mathrm{Aut(Q)},$$

i.e. a map σ such that

$\sigma_{t+t'} = \sigma_t \circ \sigma_{t'}$ and $p_{\sigma_t}(x)$ is a continuous function of t.

Thus, it is a continuous one-parameter group, i.e. curve in Aut(Q).

We have seen that an automorphism σ corresponds to a unitary or anti-unitary operator. Anti-unitary operators actually occur as symmetries, for instance in time reversal. However, for the above representation only unitary operators u_t corresponding to the symmetry σ_t can occur, since $u_t = u_{t/2}{}^2$.

It follows that the evolving state p_{σ_t} corresponds to the density operator $w_t = u_t^{-1} w u_t$.

By Stone's Theorem,

$$u_t = e^{-i/\hbar Ht}$$

so
$$w_t = e^{i/\hbar Ht} w\, e^{-i/\hbar Ht}$$

Differentiating,

$$\partial_t w_t = -i/\hbar\, [\, H, w_t\,].$$

This is the Liouville-von Neumann Equation.

Conversely, this equation yields a continuous representation of $\mathbb{R}$ into Aut(Q($\mathcal{H}$)).
For $w = P_\psi$, a pure state, this equation reduces to the customary Schrödinger Equation:

$$\partial_t \psi = -i/\hbar\, H\psi.$$

Combined Systems.

The direct sum $B_1 \oplus B_2$ of two σ-algebras (also called the free product or co-product) is well-known. (See S. Koppelberg, " Handbook of Boolean Algebras", vol.I, Chap. 4,Sect.11, Chap. 5, Sect.12).

Classical Mechanics B(Ω): $B(\Omega_1) \oplus B(\Omega_2) \simeq B(\Omega_1 \text{ x } \Omega_2)$.

General Theory Q: The *direct sum* $Q_1 \oplus Q_2$ of two σ-complexes Q_1 and Q_2 is the σ-complex consisting of all the sub-σ-algebras of the direct sums $B_1 \oplus B_2$ of all the pairs σ-algebras B_1 and B_2 contained in Q_1 and Q_2 respectively.

If S_1 and S_2 are two systems with σ-complexes Q_1 and Q_2, then the system $S_1 + S_2$ has the σ-complex $Q_1 \oplus Q_2$.

Quantum Mechanics: $Q(\mathcal{H}_1) \oplus Q(\mathcal{H}_2) \simeq Q(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

This construction of the direct sum generalizes in an obvious way from two to the direct sum of an arbitrary number of σ-complexes.

	General Mech.	CM	QM
Properties	σ-complex $Q = \cup B$	σ-algebra $B(\Omega)$	σ-complex $Q(\mathcal{H}\}$
States	$p: Q \longrightarrow [0,1]$ $p\|B$, probability measure	$p: B(\Omega) \longrightarrow [0,1]$ prob. meas.	$A: \mathcal{H} \longrightarrow \mathcal{H}$ Density operator $p(x)=tr(Ax)$
Pure States	Extreme point of convex set	$p(s) = 1 \quad \omega \epsilon s$ $= 0 \quad \omega \notin s$ i.e. $\omega \in \Omega$	Rank 1 operator i.e.unit $\phi \epsilon \mathcal{H}$ $p(x) = \langle x, x\phi \rangle$
Observables	$u: B(\mathbb{R}) \longrightarrow Q$ homomorphism	$f: \Omega \longrightarrow \mathbb{R}$ Borel function	$A: \mathcal{H} \longrightarrow \mathcal{H}$ Hermitean operator
Symmetries	$\sigma: Q \longrightarrow Q$ automorphism	$h: \Omega \longrightarrow \Omega$ canonical transformation	$U: \mathcal{H} \longrightarrow \mathcal{H}$ unitary or anti-unitary operator $\sigma(x)=UxU^{-1}$
Dynamics	$\sigma: \mathbb{R} \longrightarrow Aut(Q)$ representation	Liouville equation. $\partial_t \rho = -\{H, \rho\}$	Von Neumann-Liouville equ. $\partial_t w_t = -i\ [H, w_t]$
Conditioned States	$p(x) \longrightarrow p(x\|y)$ $x,y \in B$ in Q $p(x\|y)=p(x.y)/p(y)$	$p(x) \longrightarrow p(x\|y)$ $=p(x.y)/p(y)$	$w \longrightarrow ywy / tr(wy)$ von Neumann -Lüder's Rule
Combined Systems	$Q_1 \oplus Q_2$ Direct sum	$\Omega_1 \times \Omega_2$ Direct product	$\mathcal{H}_1 \otimes \mathcal{H}_2$ Tensor product

The Einstein-Podolsky-Rosen Experiment

By Born's Rule, if Ω is an eigenstate of A, so that $A\,\Omega = \lambda_k\,\Omega$, say, then the observable A has probability 1 of giving the value λ_k and being in the state $P_k\,\Omega$ *after the measurement of A.* Thus, the observable A is certain to give the value λ_k if it is measured. Nothing is said about the value of A before the measurement, and in particular that A already has the value λ_k. If an eclipse is certain to happen tomorrow, we do not say that it has already happened.

We will discuss the EPR experiment in the Bohm form of two spin ½ particles in the singlet state Γ of total spin 0. Suppose that in that state the two particles are separated and the spin component s_z of particle 1 is measured in some direction z. That means that the observable
$s_z \otimes I$ of the combined system is being measured.

Let $P_z^{\pm} = ½I \pm s_z$. We have the spectral decomposition

$$s_z \otimes I = (½)P_z^{+} \otimes I + (-½)P_z^{-} \otimes I$$

so the interaction algebra is $B_1 = \{ P_z^{+} \otimes I, P_z^{-} \otimes I, 1, 0\}$
We may write the singlet state

$\Gamma = \sqrt{½}(\psi_z^{+} \otimes \psi_z^{-} - \psi_z^{-} \otimes \psi_z^{+})$, where $P_z^{\pm}\psi_z^{\pm} = \psi_z^{\pm}$. $P_z^{+} \otimes I(\Gamma) = \sqrt{½}\,(\psi_z^{+} \otimes \psi_z^{-})$
$P_z^{-} \otimes I(\Gamma) = \sqrt{½}\,(\psi_z^{-} \otimes \psi_z^{+})$.

By Born's Rule,

a measurement of $s_z \otimes I$, will yield a value of ½ with probability ½ and a new state $(\psi_z^+ \otimes \psi_z^-)$ and

a value of -½ with probability ½ and new state $(\psi_z^- \otimes \psi_z^+)$.

Now, $(\psi_z^+ \otimes \psi_z^-)$ and $(\psi_z^- \otimes \psi_z^+)$ are eigenstates of $I \otimes s_z$ with respective eigenvalues -½ and ½.

By Born's Rule

the z-spin component of particle 2, **if measured**, is certain to have the opposite z-component of spin.

$I \otimes s_z$ has the spectral decomposition , $I \otimes s_z = (½)I \otimes P_z^+ + (-½)I \otimes P_z^-$, so the interaction algebra of a measurement of particle 2 is the Boolean algebra $B_2 = \{ I \otimes P_z^+, I \otimes P_z^-, 1, 0\}$.

The interaction algebra B_1 does not contain the properties

$I \otimes P_z^+$ or $I \otimes P_z^-$.

So particle 2 has not acquired the opposite spin before it is measured, as a result of the measurement on particle 1.

The relativistically invariant desciption of the EPR experiment is that if two experimenters A_1 and A_2 measure particles 1 and 2 in the same direction of spin, then an experimenter A_3 in the common part of the future light cones of A_1 and A_2 will find that the spins are in opposite directions.

Now let us consider the correlations of spin components in the z and x directions in the EPR experiment. These can be written as

$s_z \otimes I = ½ \leftrightarrow I \otimes s_z = -½$ (i.e. $P_z^+ \otimes I \leftrightarrow I \otimes P_z^-$) (1)

$s_x \otimes I = ½ \leftrightarrow I \otimes s_x = -½$ (i.e. $P_x^+ \otimes I \leftrightarrow I \otimes P_x^-$) (2)

Checking (1) by measuring $s_z \otimes I$ and $I \otimes s_z$ and seeing that they have opposite spins, precludes measuring (2), since $s_z \otimes I$ and $s_x \otimes I$ do not commute.

However, projection (1) commutes with projection (2), so we can simultaneously measure (1) and (2).

In fact, projection (1) = $1-S_z^2$ (i.e. $S_z = 0$)

projection (2) = $1-S_x^2$ (i.e. $S_x = 0$)

So, measuring S_z^2 and S_x^2 suffices to check (1) and (2).

Furthermore,

S = 0 is the same projection as the conjunction (S_z = 0).(S_x = 0), and hence is the same projection as

($s_z \otimes \mathrm{I} = ½ \leftrightarrow \mathrm{I} \otimes s_z = -½$) . ($s_x \otimes \mathrm{I} = ½ \leftrightarrow \mathrm{I} \otimes s_x = -½$)

So, measuring S allows us to to check whether both correlations (1) and (2) are true.

The Measurement Problem.

We recall the standard description of an ideal measurement.

A measurement of observable A of a system in an eigenstate ϕ_i of A, with

$$\phi_k \otimes \psi_0 \longrightarrow \phi_k \otimes \psi_k$$

By linearity, if $\phi = \Sigma a_i \phi_i$

$$\phi \otimes \psi_0 \longrightarrow \Sigma a_i \phi_i \otimes \psi_i$$

On the other hand, the completed measurement (or the prepared state), yields a state ϕ_k as indicated by the apparatus state ψ_k, so that the state of the total system is $\phi_k \otimes \psi_k$, in contradiction to the evolved state $\Sigma\, a_i \phi_i \otimes \psi_i$.

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Kolmogorov, “ Foundations of the Theory of Probability” 1933

Sample space of outcomes: s_1 , s_2, s_3 , …Boolean σ-algebra B of *events* consists of the family of subsets of sample space with operations of complement and intersection

Probability theory is based on a probability measure, i.e. a function

$$p: B \longrightarrow [0,1]$$

with domain B a σ-algebra, such that

$p(1) = 1$

$p(\cup a_i) = \Sigma p(a_i)$

for a_1, a_2, … in B such that $a_i . a_j = 0$. $i \neq j$

The outcomes are in 1-1 correspondence with measured properties P_1 , P_2 , P_3 … of the system, giving an isomorphism of the Boolean σ-algebra they generate with the event algebra.

Now, a calculation shows that (1) and (2) are projections which commute. One way to measure (1) is to measure $s_z \otimes I$ and $I \otimes s_z$ and check that they have opposite spins. However, this precludes one checking (2) by measuring $s_x \otimes I$ and $I \otimes s_x$ since $s_z \otimes I$ and $s_x \otimes I$ do not commute. However, a calculation shows that *projection (1) is identical to the projection* $1 - S_z^2$ *(i.e.the property* $S_z = 0$) *and (2) is equally the projection* $1 - S_x^2$ *(i.e.* $S_x = 0$*).* Since S_z^2 and S_x^2 commute we can measure both and check whether (1) and (2) are simultaneously true. In fact, a calculation shows that *the conjunction* $S_z = 0 \,.\, S_x = 0$ *is equal to the projection* $S = 0$. So a measurement of S suffices to check whether (1) and (2) are simultaneously true. Of course, we would not determine the values of the spin components in either the z or x directions this way, but that is the whole point of this discussion: that a compound statement such as a correlation, can have a truth value without the component statements having a value. Just as a v b may be certain even though a and b are not, correlations (1) and (2) may be certain even when $s_z \otimes I$ and $s_x \otimes I$ are not.

Claim this are reverting to the classical notion of intrinsic properties. The spin components are extrinsic properties of each particle, which do not have values until the appropriate interaction. We emphasize that this is not only a necessary consequence of our interpretation, but follows from a careful application of standard quantum mechanical principles, namely that a system in an eigenstate of an observable will yield the eigenvalue as the value of that observable only if and when that observable is measured.

We have not in this discussion mentioned a word about special relativity. Indeed the spin EPR phenomenon has nothing to do with position or motion and is independent of relativity. However, EPR with space-like separated particles has been used to put in question the full Lorentz invariance of of quantum mechanics. This is replaced by a weaker notion that EPR correlations cannot be used for faster than light signaling. We believe that Lorentz invariance is a fundamental symmetry principle which gives rise to fundamental observables, and is not simply an artifact of signaling messages between agents. As we just saw, the measurement of spin for one particle has absolutely no effect on the spin of the other. The relativistically invariant description of the EPR experiment is that if experimenters A_1 and A_2 measure particles 1 and 2, and the directions of spin in which they are measured are the same, then an agent B in the common part of the future light cones of A_1 and A_2 will find that the spins are in opposite directions.

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To be sure, the correlations of spin in EPR are quite different from classical correlations. The standard example of a pair of gloves, in which a knowledge that I have that the glove I have is left-handed tells me immediately that the glove I left in Australia is right-handed is certainly different from EPR pair. In the latter, we obtain correlations in arbitrary directions of spins that did not have values before they are measured. This is due to the difference between extrinsic and intrinsic properties.

We analyze this in more detail. In our triple experiment on a spin 1 system, the total spin angular momentum S is conserved, with value $\sqrt{2}$. Hence,

$S^2 = S_x^2 + S_y^2 + S_z^2 = 2$, for any ortho-frame (x,y,z). We may equivalent write this as

$S_z^2 = 0$ if and only if $S_x^2 = 1$ and $S_y^2 = 1$, or equally $S_z^2 = 1$ if and only if $S_x^2 = 0$ or $S_y^2 = 0$ (*)

This condition is true independently of whether S_x^2, S_y^2, and S_z^2 have values. It is erroneous to think that extrinsic properties, or more generally observables, stop existing when they do not have values,(any more than an operator stops existing when the state is not an eigenstate of the operator). Indeed, the σ-complex Q is constructed out of properties, of which only a current interacting σ-algebra has values.

The above condition subsists because of quantum laws about angular momentum which are independent of time or current interactions. This is a general phenomenon of σ-complexes of extrinsic properties. An element c in Q may lie in two σ-algebras B and B', and thus be a union

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c = a v b in B and a union c = a' v b', where a,b,a',b' are atoms in Q. In the example above, we have (*), i.e. $S_z^2 = 1-S_x^2 + 1-S_y^2$,

or $S_z^2 = 1$ if and only if $S_{x'}^2 = 0$ or $S_{y'}^2 = 0$,

i.e. $S_z^2 = 1 - S_{x'}^2 + 1-S_{y'}^2$

These relations exist in Q independently of whether B or B' have truth values assigned.

In the EPR experiment, the total spin S of the two particles is conserved as they are separated, and so S = 0 remains true. Now the relation S = 0 implies that $s_z^1 = ½$ if and only if $s_z^2 = -½$, for any direction z.

Again, this relation is a consequence of quantum laws about angular momentum, and is not dependent on any particular interaction. The measurement of s_z^1, i.e. the observable $s_z^1 \otimes I$, yields the interaction algebra $\{0,1,s_z^1=½,s_z^1=-½\}$. As we have seen, the resulting state is an eigenstate of the observable $I \otimes s_z^2$, so that all we can say is that particle 2 has a spin value of ½ or -½ if measured, but not that it has a value as a result of the measurement of particle S_1.

In summary, the extrinsic properties of a σ-complex may have relations subsisting among its elements because of general laws of physics, such as conservation laws, which are timeless and independent of particular interactions. (We shall discuss such laws in the section on Symmetries). This fact allows to interpret the EPR phenomenon in a fully relativistically invariant way.

Now let us consider the correlations of spin components in the z and x directions in the EPR experiment. These can be written as

$s_z \otimes I = ½ \leftrightarrow I \otimes s_z = -½$ (i.e. $P_z^+ \otimes I \leftrightarrow I \otimes P_z^-$) (1)

$s_x \otimes I = ½ \leftrightarrow I \otimes s_x = -½$ (i.e. $P_x^+ \otimes I \leftrightarrow I \otimes P_x^-$) (2)

Now, a calculation shows that (1) and (2) are projections which commute. One way to measure (1) is to measure $s_z \otimes I$ and $I \otimes s_z$ and check that they have opposite spins. However, this precludes one checking (2) by measuring $s_x \otimes I$ and $I \otimes s_x$ since $s_z \otimes I$ and $s_x \otimes I$ do not commute. However, a calculation shows that *projection (1) is identical to the projection 1-S_z^2 (i.e.the property $S_z = 0$)* *and (2) is equally the projection 1- S_x^2 (i.e. $S_x = 0$).* Since S_z^2 and S_x^2 commute we can measure both and check whether (1) and (2) are simultaneously true. In fact, a calculation shows that *the conjunction $S_z = 0$. $S_x = 0$ is equal to the projection S = 0.* So a measurement of S suffices to check whether (1) and (2) are simultaneously true. Of course, we would not determine the values of the spin components in either the z or x directions this way, but that is the whole point of this discussion: that a compound statement such as a correlation, can have a truth value without the component statements having a value. Just as a v b may be certain even though a and b are not, correlations (1) and (2) may be certain even when $s_z \otimes I$ and $s_x \otimes I$ are not.

This way of understanding the EPR correlations for non-commutative spins is not an ad hoc approach to this particular experiment. In [6] we show that any correlation between two systems can be understood in a like manner.

We shall discuss the EPR phenomenon in the Bohm form of a system S_1+S_2 of two spin ½ particles in the singlet state Γ of total spin 0. Suppose that in that state the two particles are separated and the spin component s_z of particle S_1 (i.e. the observable $s_z \otimes I$ of the system
S_1+S_2 is measured in some direction z. Let $\phi_z^{\pm}$ be the eigenstates of S_1 belonging to the eigenvalues $\pm$½ of s_z , and $\psi^{\pm}$ those of S_2 .

The singlet state of S_1+S_2 is

$\sqrt{½}(\phi_z^{+} \otimes \psi_z^{-} - \phi_z^{-} \otimes \psi_z^{+})$.

If particle S_1 is measured to be, say, spin up, the state of S_1+S_2 is then

$\phi_z^{+} \otimes \psi_z^{-}$.

In our notation, if p(x) is the singlet state, then the new state is the conditional state

$p(x_{\_}| s_z \otimes I = ½)$.

This state is an eigenstate of the observable $I \otimes s_z$ belonging to the eigenvalue -½ , so that

$p(I \otimes s_z = - ½| s_z \otimes I = ½) = 1$.

This means that if the component of spin of particle S_2 is measured, then it is certain to have the opposite spin to particleS_1. It does *not* mean that particle S_2 has acquired the opposite spin before it is measured, as a result of the measurement on particle S_1. Those who

Let $P_z^{\pm} = ½I \pm s_z$. We have the spectral decomposition

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By Born's Rule, a measurement of $s_z \otimes I$, the z-component of spin of particle 1, will yield the a value of ½ with probability ½ and a new state $(\psi_z^{+} \otimes \psi_z^{-})$, or a value of -½with probability ½ and new state $(\psi_z^{-} \otimes \psi_z^{+})$. Now, $(\psi_z^{+} \otimes \psi_z^{-})$ and $(\psi_z^{-} \otimes \psi_z^{+})$ are eigenstates of $I \otimes s_z$ with respective eigenvalues -½ and ½. It follows again from Born's Rule that if the z-spin component of particle 2 is measured, then it is certain to have the opposite z-component of spin.

Now, $I\otimes s_z$ has the spectral decomposition , $I\otimes s_z = (½)I\otimes P_z^+ + (-½)I\otimes P_z^-$, so the interaction algebra of a measurement of particle 2 is the Boolean algebra $B_2 = \{ I\otimes P_z^+, I\otimes P_z^-, 1, 0\}$.

Our conclusion is that if particle 1 is measured and its spin is, say, ½ then *if particle 2 is measured,* it is certain to have the value ½. It does *not* mean that particle 2 has acquired the opposite spin before it is measured, as a result of the measurement on particle 1. Those who claim this are reverting to the classical notion of intrinsic properties. The interaction algebra B_1 does not contain the properties $I\otimes P_z^+$, $I\otimes P_z^-$. The spin components do not have values unless and until the appropriate interaction happens.

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