No Go Theorems 1: Don’t Even Go There!

Solstice of Foundations, ETH Zurich

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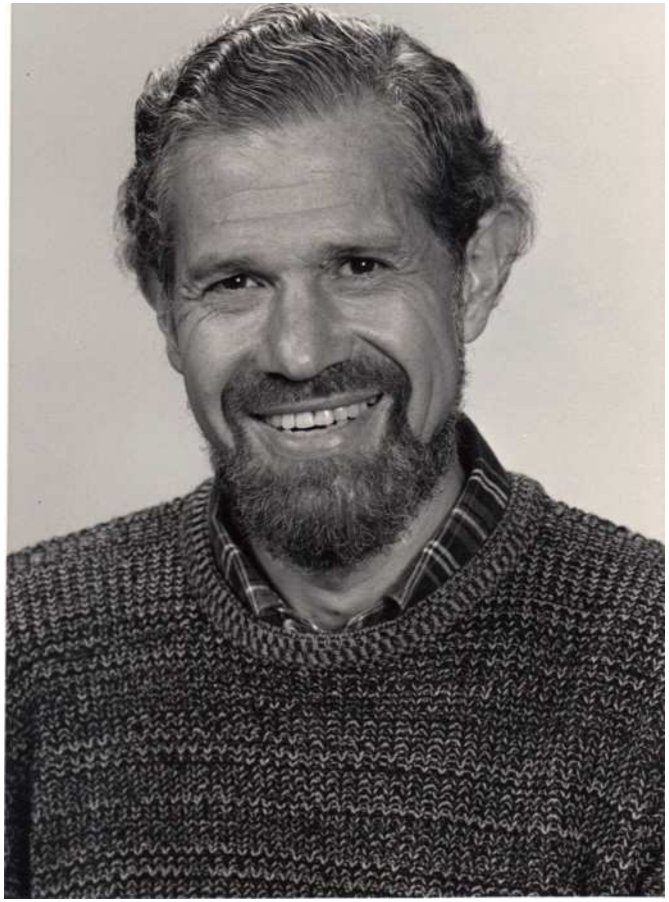
Introduction

Source: `http://learn-math.info`

. . . long may Louis de Broglie continue to inspire those who suspect that what is proved by impossibility proofs is lack of imagination.[a]

[a] J. Bell, "On the impossible pilot wave", *Speakable and Unspeakable in Quantum Mechanics*, 2nd ed. pp. 159–168 (CUP, 2004)

Simon Kochen



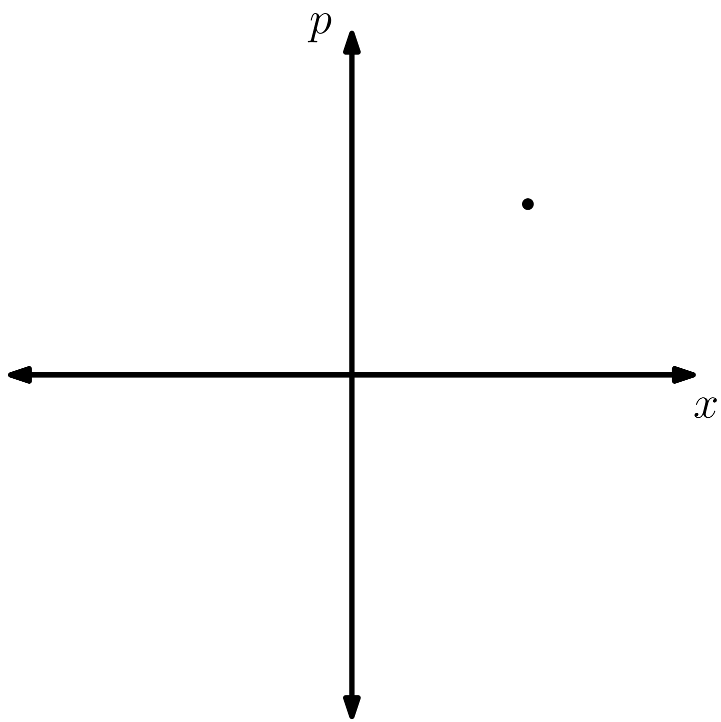
Source: `http://www.helixcenter.org`

Ernst Specker

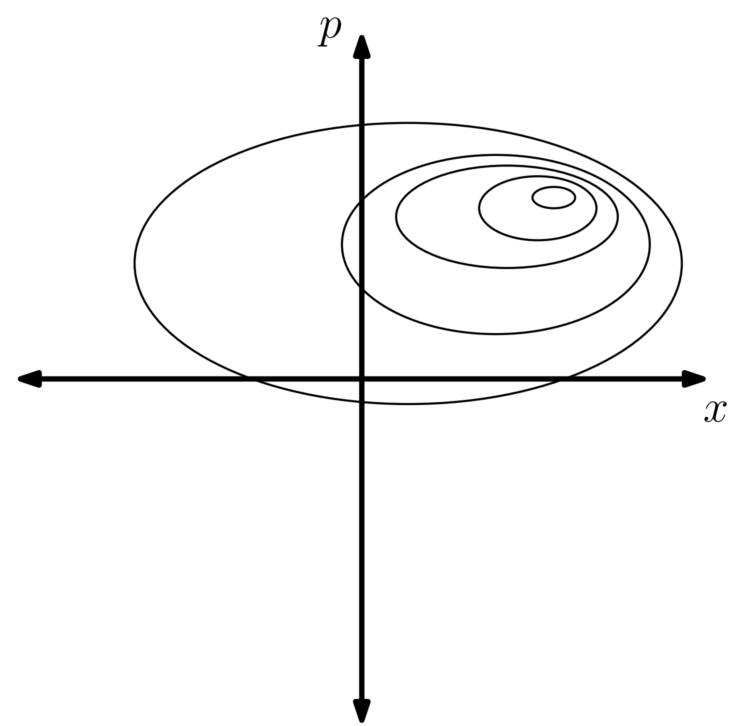
Source: `https://en.wikipedia.org`

Classical states

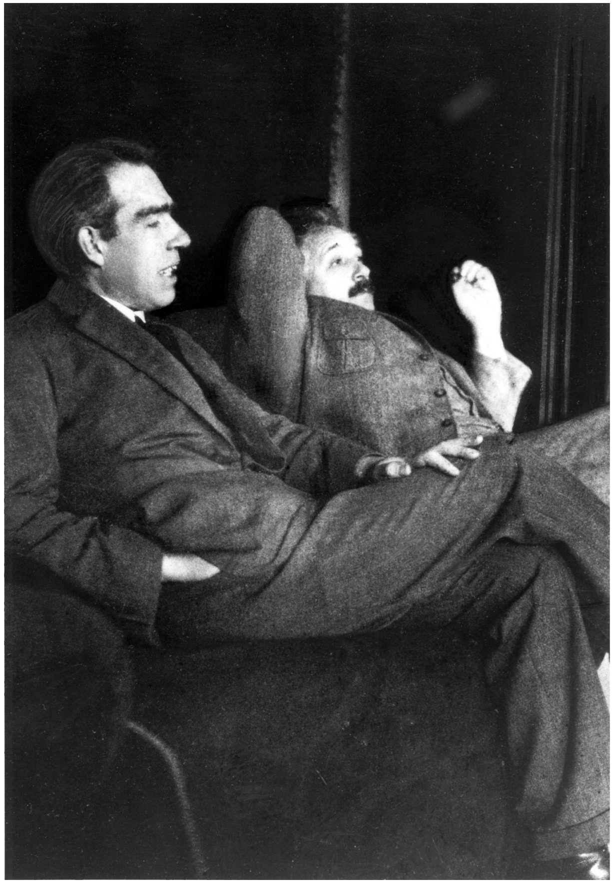
Ontic state



Epistemic state



Bohr and Einstein: ψ-epistemicists



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> There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature. — Niels Bohr[a]

> [t]he ψ-function is to be understood as the description not of a single system but of an ensemble of systems. — Albert Einstein[b]

[a] Quoted in A. Petersen, "The philosophy of Niels Bohr", *Bulletin of the Atomic Scientists* Vol. 19, No. 7 (1963)

[b] P. A. Schilpp, ed., *Albert Einstein: Philosopher Scientist* (Open Court, 1949)

Penrose: ψ-ontologist

It is often asserted that the state-vector is merely a convenient description of ‘our knowledge’ concerning a physical system—or, perhaps, that the state-vector does not really describe a single system but merely provides probability information about an ‘ensemble’ of a large number of similarly prepared systems. Such sentiments strike me as unreasonably timid concerning what quantum mechanics has to tell us about the *actuality* of the physical world. — Sir Roger Penrose[1]

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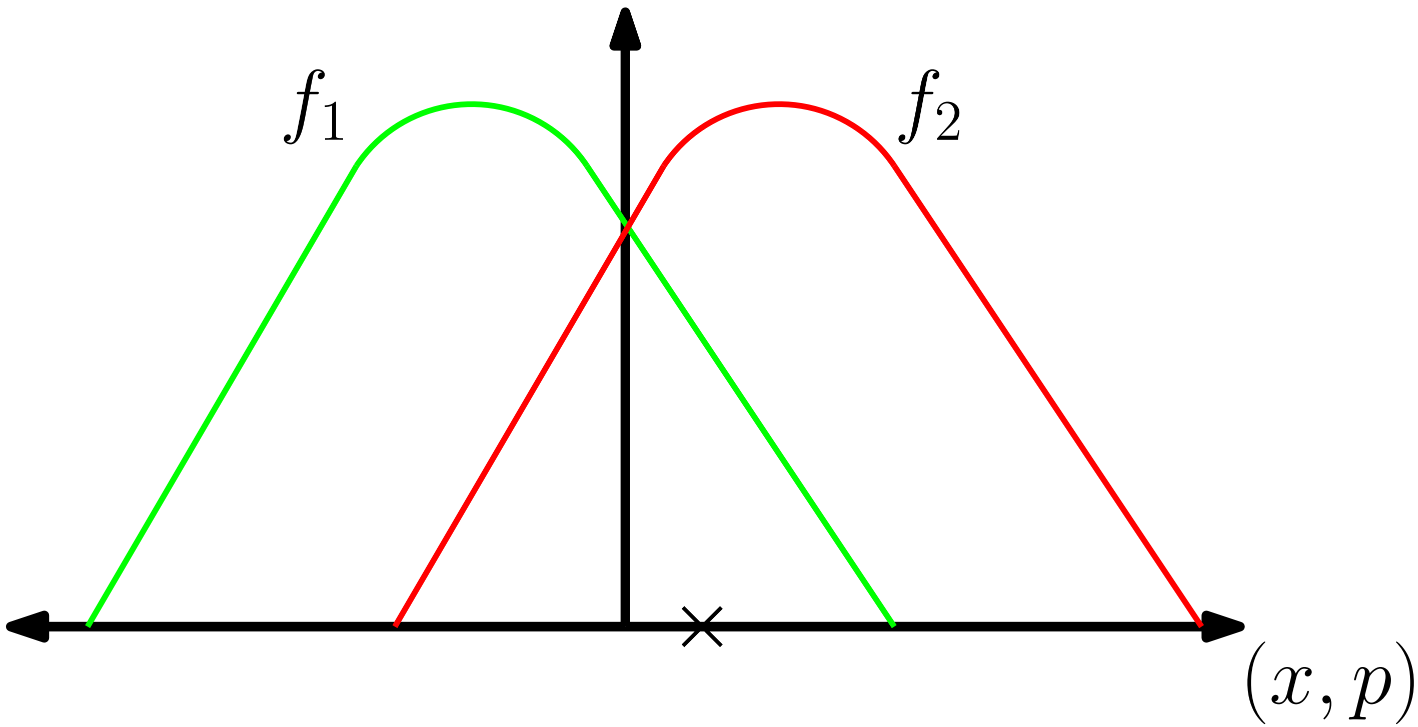
[1] R. Penrose, *The Emperor’s New Mind* pp. 268–269 (Oxford, 1989)

Interpretations of quantum theory

		ψ-epistemic	ψ-ontic
Copenhagenish	Conventional	Copenhagen Jeff Bub Healy's Quant. Pragmatism	
	Perspectival	QBism Rovelli's "Relational" QM Perspectival Copenhagen	
Realist	Ontic Model		Dirac-von Neumann Bohmian mechanics Spontaneous collapse Modal interpretations
	Exotic	Retrocausality Ironic many worlds	Everett/Many worlds

Epistemic states overlap



Lecture 1:

- Recap of Ontological Models
- The Pusey-Barrett-Rudolph Theorem

Lecture 2:

- Overlap Bounds
- Ontological Excess Baggage
- Relationships Between No-Go Theorems

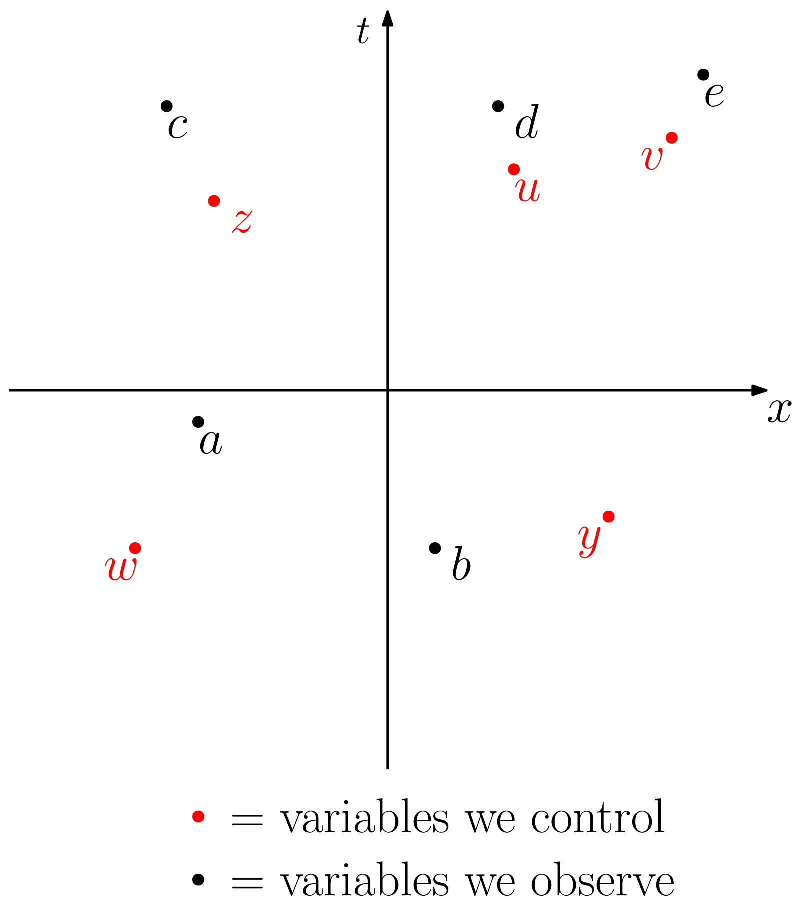
References

- Good references for most of the material in these lectures are:

  - D. Jennings and ML, “No Return to Classical Reality”, Contemp. Phys. 57:60–82, arXiv:1501.03202 (2016).
  - ML, “Is the quantum state real? An extended review of ψ-ontology theorems”, Quanta 3:67–155, arXiv:1409.1570 (2014).

Ontological Models

- Operational Theories:



Theory predicts $\text{Prob}(a, b, c, d, e | u, v, w, y, z)$

- Ontic States:
- In addition to the variables we control and observe, there may be additional physical properties λ taking values in a set Λ.
  - λ is called an *ontic state*.
  - Λ is the *ontic state space*.

- *(Single World) Realism*: On each run of the experiment, the operational variables and λ each take a definite value.

- *Independence*: Each run of the experiment is independent and identically distributed.

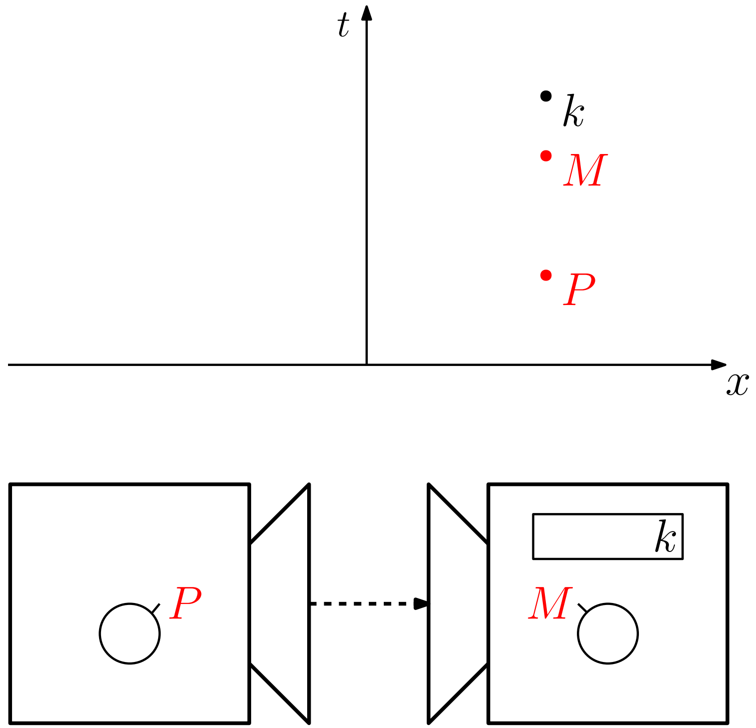
$$\Leftrightarrow \qquad \exists \text{ a joint probability distribution}$$

$$\text{Pr}(a, b, c, d, e, \lambda | u, v, w, y, z).$$

- Note: We use Prob for operational theory probabilities and Pr for probabilities involving ontic states.

- The model *reproduces the operational predictions* if

$$\text{Prob}(a, b, c, d, e | u, v, w, x, y, z) = \int_\Lambda \text{Pr}(a, b, c, d, e, \lambda | u, v, w, y, z) \mathrm{d}\lambda.$$



- We focus on a simple kind of experiment:
  - P is a choice of *preparation*.
  - M is a choice of *measurement*.
  - k is the *outcome* of th measurement.

- In a model that reproduces the operational predictions, we have

$$\text{Prob}(k|P,M) = \int_\Lambda \text{Pr}(k,\lambda|P,M)\text{d}\lambda.$$

- In general. we can write

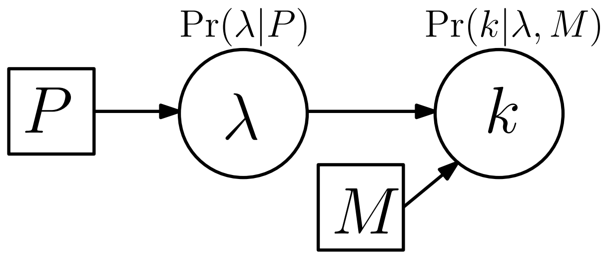
$$\text{Prob}(k|P,M) = \int_\Lambda \text{Pr}(k,\lambda|P,M)\mathrm{d}\lambda$$
$$= \int_\Lambda \text{Pr}(k|\lambda,P,M)\text{Pr}(\lambda|P,M)\mathrm{d}\lambda.$$

- To get an ontological model, we impose two more assumptions:
  - *Measurement independence*: $\text{Pr}(\lambda|P,M) = \text{Pr}(\lambda|P)$.
  - *λ-mediation*: $\text{Pr}(k|\lambda,P,M) = \text{Pr}(k|\lambda,M)$.
- So, we have

$$\text{Prob}(k|P,M) = \int_\Lambda \text{Pr}(k|\lambda,M)\text{Pr}(\lambda|P)\mathrm{d}\lambda.$$

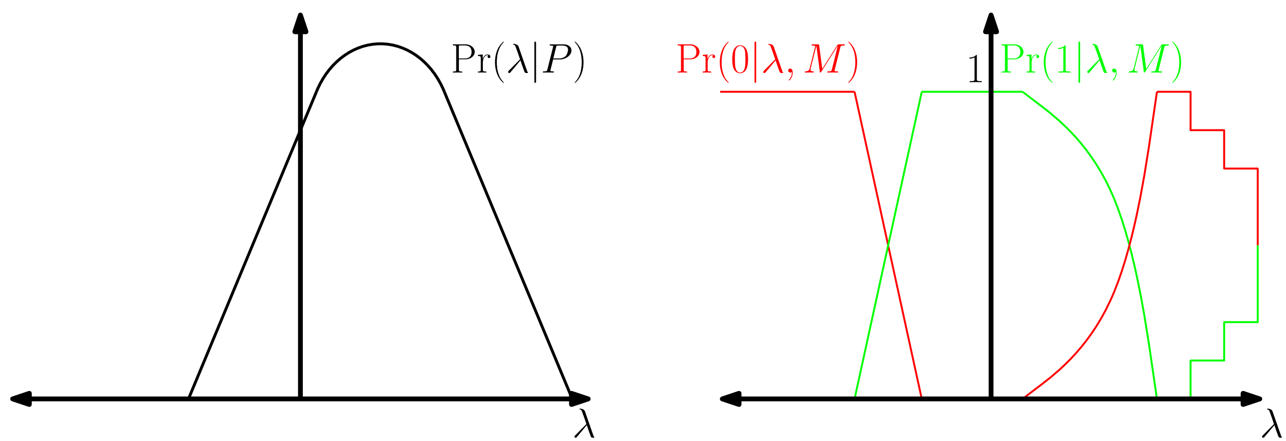
- A model with ontic states satisfying (signle world) realism, independence, measurement independence, and λ-mediation is called an *ontological model*.

- Alternatively, an ontological model has the following causal structure



$$\text{Pr}(\lambda, k|P, M) = \text{Pr}(k|\lambda, M)\text{Pr}(\lambda|P)$$

- $\text{Pr}(\lambda|P)$ is called the *epistemic state*.
- $\text{Pr}(k|\lambda, M)$ is called the *response function* of the measurement.



$\int_\Lambda \text{Pr}(\lambda|P)d\lambda = 1$ $\qquad$ $\sum_k \text{Pr}(k|\lambda, M) = 1$

- We are most interested in the case where the operational theory has a model within quantum theory, in which case:
- Each preparation P is assigned a density operator ρ_P.
- Each measurement M is assigned a POVM $\{E_k^M\}$, s.t.

$$\sum_k E_k^M = I.$$

- The operational probabilities are given by

$$\text{Prob}(k|P, M) = \text{Tr}\left(E_k^M \rho_P\right).$$

- and so an ontological model must satisfy

$$\text{Tr}\left(E_k^M \rho_P\right) = \int_\Lambda \text{Pr}(k|\lambda, M)\text{Pr}(\lambda|P)\text{d}\lambda.$$

- The mappings $P \to \rho_P$ and $(M, k) \to E_k^M$ need not be one-to-one.
  - $\rho_{P_1} = \rho_{P_2}$ does not imply Pr$(\lambda|P_1) =$ Pr$(\lambda|P_2)$.
  - $E_{k_1}^{M_1} = E_{k_2}^{M_2}$ does not imply Pr$(k_1|\lambda, M_1) =$ Pr$(k_2|\lambda, M_2)$.
- In fact, in general, they cannot be because of contextuality.
- It is very naughty to write:
  - Pr$(\lambda|\rho)$ instead of Pr$(\lambda|P)$,
  - Pr$(k|\lambda, E)$ instead of Pr$(k|\lambda, M)$.
- However, we will often do so to avoid clutter.
  - A statement involving Pr$(\lambda|\rho)$ really means:

$$\forall P \quad \text{s.t.} \quad \rho_P = \rho, \ \text{the same statement for Pr}(\lambda|P).$$

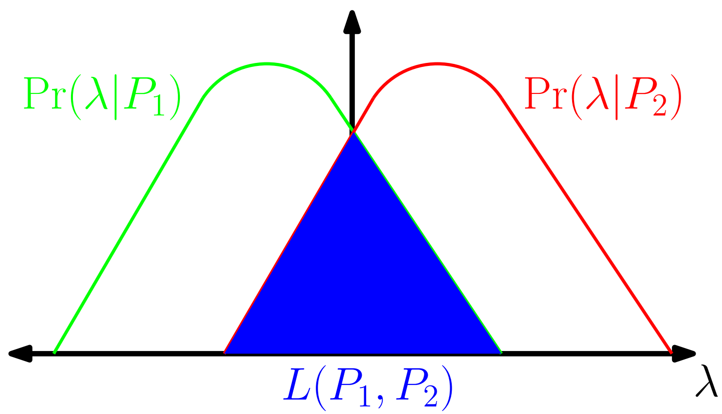
  - A statement involving Pr$(k|\lambda, E)$ really means:

$$\forall (M, k) \quad \text{s.t.} \quad E_k^M = E, \ \text{the same statement for Pr}(k|\lambda, M).$$

ψ-Ontology

- We introduce a measure of *epistemic overlap* in an ontological model:

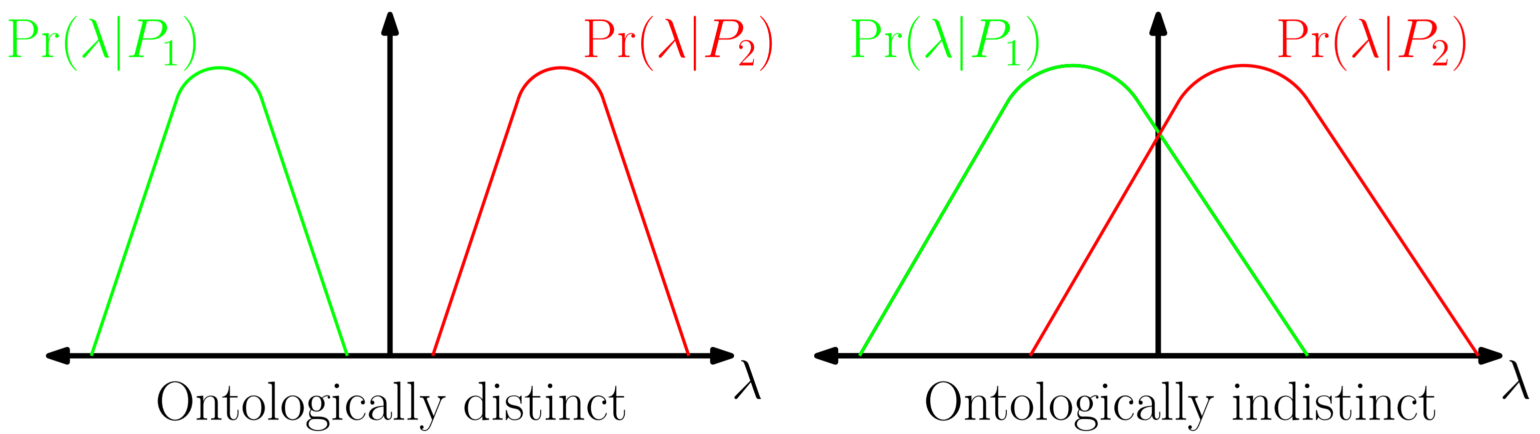
$$L(P_1, P_2) = \int_\Lambda \min\{\Pr(\lambda|P_1), \Pr(\lambda|P_2)\} d\lambda$$



- We will also use the n-way overlap:

$$L(\{P_j\}_{j=1}^n) = \int_\Lambda \min\{\Pr(\lambda|P_j)\}_{j=1}^n d\lambda$$

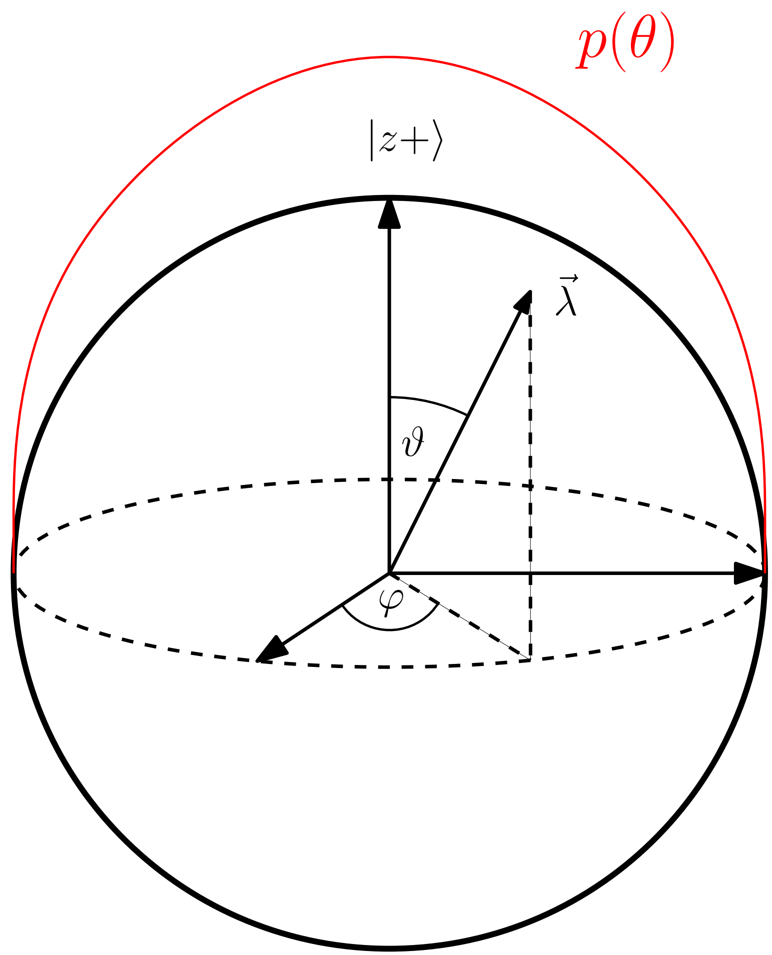
- P_1 and P_2 are *ontologically distinct* in an ontological model if $L(P_1, P_2) = 0$.



- An ontological model of quantum theoy is *ψ-ontic* if every pair of preparations corresponding to distinct pure states is ontologically distinct. Otherwise it is *ψ-epistemic*.
- Naughty notation: $L(\psi_1, \psi_2) = 0$ means:

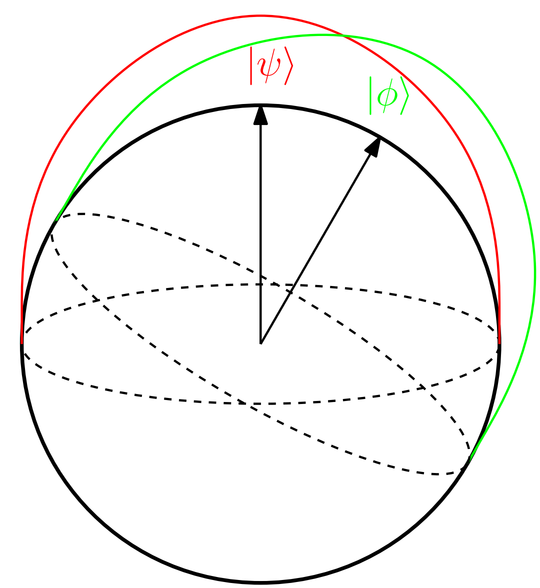
$$\forall\ P_1, P_2 \text{ s.t. } \rho_{P_1} = |\psi_1\rangle\langle\psi_1| \text{ and } \rho_{P_2} = |\psi_2\rangle\langle\psi_2|\,,\ \ L(P_1, P_2) = 0.$$

The Kochen-Specker model for a qubit



$$\mu_{z+}(\Omega) = \int_{\Omega} p(\vartheta)\sin\vartheta d\vartheta d\varphi$$

$$p(\vartheta) = \begin{cases} \frac{1}{\pi}\cos\vartheta, & 0 \leq \vartheta \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \vartheta \leq \pi \end{cases}$$



S. Kochen and E. Specker, *J. Math. Mech.*, 17:59–87 (1967)

Models for arbitrary finite dimension

- Lewis et. al. provided a ψ-epistemic model for all finite d.
  - P. G. Lewis et. al., *Phys. Rev. Lett.* 109:150404 (2012) arXiv:1201.6554
- Aaronson et. al. provided a similar model in which every pair of nonorthogonal states is ontologically indistinct.
  - S. Aaronson et. al., *Phys. Rev. A* 88:032111 (2013) arXiv:1303.2834
- So we can either introduce new assumptions, or prove something weaker than ψ-ontology.

The Pusey-Barrett-Rudolph Theorem

The Preparation Independence Postulate

- The PBR Theorem[2] shows that, under an additional assumption called the *Preparation Independence Postulate (PIP)*, ontological models of quantum theory must be ψ-ontic.

- The PIP can be broken down into two assumptions:
  - The *Cartesian Product Assumption*:
    When two systems are prepared independently in a product state $|\psi\rangle_A \otimes |\phi\rangle_B$, the joint ontic state space is $\Lambda_{AB} = \Lambda_A \times \Lambda_B$, i.e. each system has its own ontic state $\lambda_{AB} = (\lambda_A, \lambda_B)$.
  - The *No Correlation Assumption*:
    The epistemic state corresponding to $|\psi\rangle_A \otimes |\phi\rangle_B$ is

$$\text{Pr}(\lambda_A, \lambda_B|\psi_A, \phi_B) = \text{Pr}(\lambda_A|\psi_A)\text{Pr}(\lambda_B|\phi_B).$$

[2] Nature Physics 8:475–478 (2012).

- In general, a joint system with two subsystems may have global ontic properties that do not reduce to properties of the subsystems.
  - In a ψ-ontic model, an entangled state would be an example of such a property.
  - So, in general we need $\Lambda_{AB} = \Lambda_A \times \Lambda_B \times \Lambda_{\text{global}}$.
  - All we require from the Cartesian Product Assumption is that Λ_{global} plays no role in determining the measurement outcomes *when a product state is prepared*, e.g. for product preparations λ_{global} always takes the same value.
  - Then, the No Correlation Assumption should be read as applying to the marginal on $\Lambda_A \times \Lambda_B$.

- **Theorem**: An ontological model of quantum theory that satisfies the PIP must be ψ-ontic.

- Proof strategy: We follow a proof by C. Moseley[3].

1. Prove that $|\psi_1\rangle$ and $|\psi_2\rangle$ are ontologically distinct when $|\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{2}$ using *antidistinguishability*.
2. Prove the case $|\langle\psi_1|\psi_2\rangle|^2 < \frac{1}{2}$ by reduction to 1.
3. Prove the case $\frac{1}{2} < |\langle\psi_1|\psi_2\rangle|^2 < 1$ by reduction to 2.

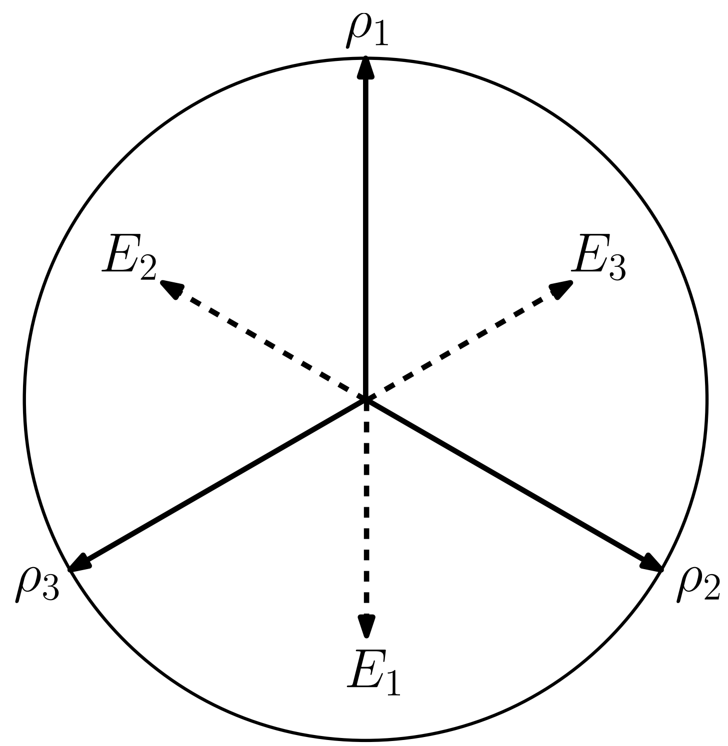
[3] arXiv:1401.0026

Antidistinguishability

- A set $\{\rho_j\}_{j=1}^n$ of n quantum states is *antidistinguishable* if there exists an n-outcome POVM $\{E_j\}_{j=1}^n$ such that

$$\forall j, \qquad \text{Tr}\left(E_j \rho_j\right) = 0.$$

- Example:



Antidistinguishability and Overlap

- **Lemma**: If a set of states $\{\rho_j\}_{j=1}^n$ is antidistinguishable then, in any ontological model that reproduces the quantum predictions

$$L(\{\rho_j\}) = 0.$$

- Proof for finite Λ:
  - $L(\{\rho_j\}_{j=1}^n) = \sum_\lambda \min_{j=1}^n \{\text{Pr}(\lambda|\rho_j)\}$ so it is > 0 iff there exists a λ for which all $\text{Pr}(\lambda|\rho_j) > 0$.
  - Suppose there exists such a λ. We require $\text{Pr}(E_j|\lambda) = 0$ for all j to reproduce the quantum predictions.
  - But $\sum_{j=1}^n \text{Pr}(E_j|\lambda) = 1$, so no such λ can exist.

- By antidistinguishability,

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \text{Tr}\left(E_k \rho_k\right) \\ &= \int_\Lambda \left[\sum_k \text{Pr}(E_k|\lambda)\text{Pr}(\lambda|\rho_k)\right] \mathrm{d}\lambda \\ &\geq \int_\Lambda \left[\sum_k \text{Pr}(E_k|\lambda) \min_{j=1}^{n}\{\text{Pr}(\lambda|\rho_j)\}\right] \mathrm{d}\lambda \\ &= \int_\Lambda \left[\sum_k \text{Pr}(E_k|\lambda)\right] \min_{j=1}^{n}\{\text{Pr}(\lambda|\rho_j)\}\mathrm{d}\lambda. \end{aligned}$$

- But $\sum_{k=1}^{n} \text{Pr}(E_k|\lambda) = 1$, so

$$0 = \int_\Lambda \min_{j=1}^{n}\{\text{Pr}(\lambda|\rho_j)\}\mathrm{d}\lambda = L(\{\rho_j\}_{j=1}^{n}).$$

- W.l.o.g. we can choose a basis such that the two states are

$$|\psi_1\rangle = |0\rangle\,, \qquad |\psi_2\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right).$$

- Now consider the four states

$$|\Psi_1\rangle = |0\rangle \otimes |0\rangle \qquad |\Psi_2\rangle = |0\rangle \otimes |+\rangle$$
$$|\Psi_3\rangle = |+\rangle \otimes |0\rangle \qquad |\Psi_4\rangle = |+\rangle \otimes |+\rangle$$

- and the orthonormal basis

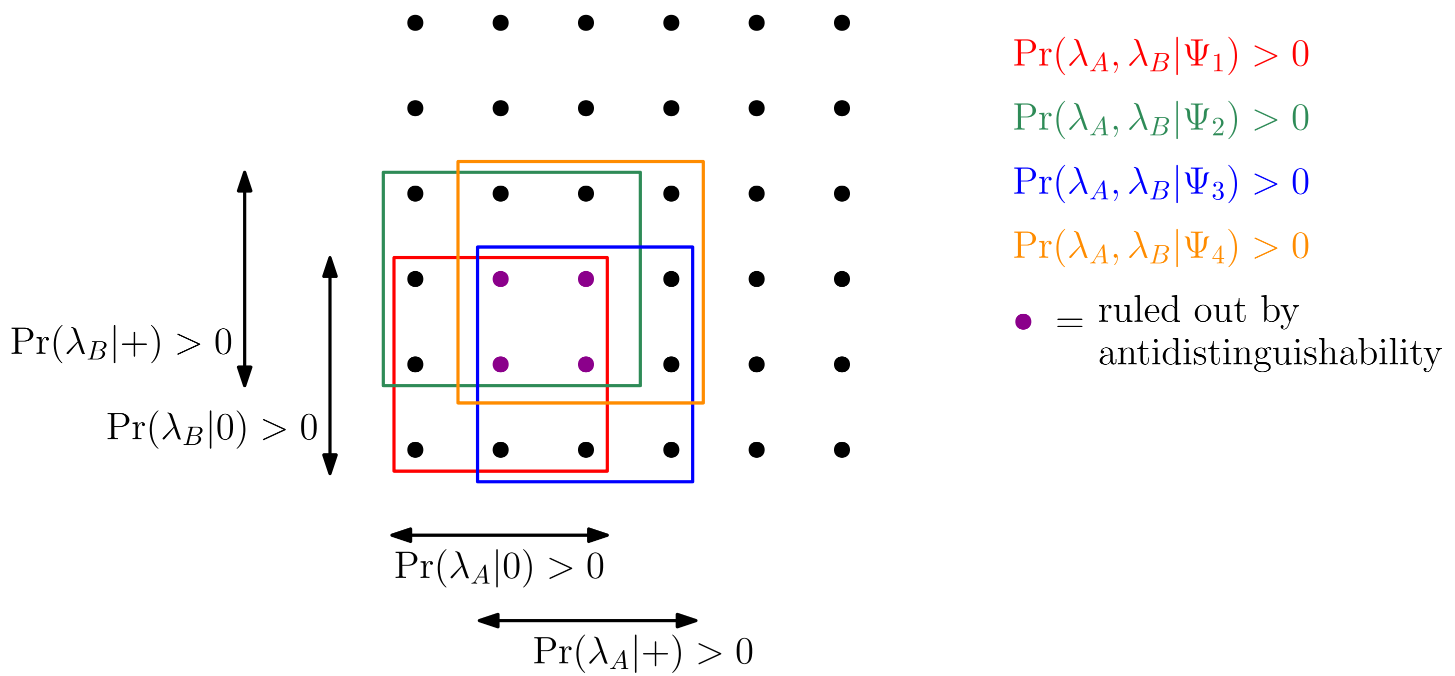
$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right)$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |-\rangle + |1\rangle \otimes |+\rangle\right)$$
$$|\Phi_3\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |1\rangle + |-\rangle \otimes |0\rangle\right)$$
$$|\Phi_4\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle\right)$$

- We have $|\langle\Phi_j|\Psi_j\rangle|^2 = 0$, so $\{\Psi_j\}_{j=1}^4$ is antidistinguishable.

- By the PIP:

$$\begin{aligned}\Pr(\lambda_A, \lambda_B|\Psi_1) &= \Pr(\lambda_A|0)\Pr(\lambda_B|0)\\ \Pr(\lambda_A, \lambda_B|\Psi_2) &= \Pr(\lambda_A|0)\Pr(\lambda_B|+)\\ \Pr(\lambda_A, \lambda_B|\Psi_3) &= \Pr(\lambda_A|+)\Pr(\lambda_B|0)\\ \Pr(\lambda_A, \lambda_B|\Psi_4) &= \Pr(\lambda_A|+)\Pr(\lambda_B|+)\end{aligned}$$

- Proof for finite Λ:



- In order to avoid having the purple ontic states, $\Pr(\lambda|0)$ and $\Pr(\lambda|+)$ must have no overlap.

- General proof:

$$
\begin{aligned}
0 = L(\{\Psi_j\}_{j=1}^4) &= \int_{\Lambda_A} \mathrm{d}\lambda_A \int_{\Lambda_B} \mathrm{d}\lambda_B \left[\min_{j=1}^{4} \{\text{Pr}(\lambda_A, \lambda_B|\Psi_j)\}\right] \\
&= \int_{\Lambda_A} \mathrm{d}\lambda_A \int_{\Lambda_B} \mathrm{d}\lambda_B \min\{\text{Pr}(\lambda_A|0)\text{Pr}(\lambda_B|0), \text{Pr}(\lambda_A|0)\text{Pr}(\lambda_B|+), \\
&\qquad \text{Pr}(\lambda_A|+)\text{Pr}(\lambda_B|0), \text{Pr}(\lambda_A|+)\text{Pr}(\lambda_B|+)\} \\
&= \left[\int_{\Lambda_A} \mathrm{d}\lambda_A \min\{\text{Pr}(\lambda_A|0), \text{Pr}(\lambda_A|+)\}\right] \\
&\qquad \times \left[\int_{\Lambda_B} \mathrm{d}\lambda_B \min\{\text{Pr}(\lambda_B|0), \text{Pr}(\lambda_B|+)\}\right] \\
&= L(0,+)^2.
\end{aligned}
$$

- **Theorem**: If $|\langle\psi_1|\psi_2\rangle|^2 < |\langle\psi_3|\psi_4\rangle|^2$ then there exists a CPT map $\mathcal{E}$ s.t.

$$\mathcal{E}(|\psi_1\rangle\langle\psi_1|) = |\psi_3\rangle\langle\psi_3|\,, \qquad \mathcal{E}(|\psi_2\rangle\langle\psi_2|) = |\psi_4\rangle\langle\psi_4|\,.$$

- So, our measurement procedure consists of mapping $|\psi_1\rangle$ to $|0\rangle$, $|\psi_2\rangle$ to $|+\rangle$, and then applying the same argument as before.
- We can always choose a basis such that

$$|\psi_1\rangle = |0\rangle\,, \qquad |\psi_2\rangle = \sin\theta\,|0\rangle + \cos\theta\,|1\rangle\,,$$

with $0 \leq \theta < \frac{\pi}{4}$.

- Then, you can check that $\mathcal{E}(\rho) = M_1\rho M_1^\dagger + M_2\rho M_2^\dagger$, with

$$M_1 = |0\rangle\langle 0| + \tan\theta\,|1\rangle\,, \quad M_2 = \sqrt{\frac{1-\tan^2\theta}{2}}\,(|0\rangle + |1\rangle)\,\langle 1|\,.$$

is CPT and does the job.

- Let $|\Psi_1\rangle = |\psi_1\rangle^{\otimes n}$ and $|\Psi_2\rangle = |\psi_2\rangle^{\otimes n}$.
- Since $|\langle\Psi_1|\Psi_2\rangle| = |\langle\psi_1|\psi_2\rangle|^n$, there exists an n such that

$$|\langle\Psi_1|\Psi_2\rangle|^2 < \frac{1}{2}.$$

- Apply the previous argument $|\Psi_2\rangle$.

$$|\Psi_1\rangle \otimes |\Psi_1\rangle \qquad |\Psi_1\rangle \otimes |\Psi_2\rangle$$
$$|\Psi_2\rangle \otimes |\Psi_1\rangle \qquad |\Psi_2\rangle \otimes |\Psi_2\rangle$$

- By the PIP,

$$\text{Pr}(\lambda_1, \lambda_2, \cdots, \lambda_n|\Psi_1) = \text{Pr}(\lambda_1|\psi_1)\text{Pr}(\lambda_2|\psi_1)\cdots\text{Pr}(\lambda_n|\psi_1)$$
$$\text{Pr}(\lambda_1, \lambda_2, \cdots, \lambda_n|\Psi_2) = \text{Pr}(\lambda_1|\psi_2)\text{Pr}(\lambda_2|\psi_2)\cdots\text{Pr}(\lambda_n|\psi_2)$$

and these have zero overlap iff $\text{Pr}(\lambda|\psi_1)$ and $\text{Pr}(\lambda|\psi_2)$ do.

Conclusion

- The PBR theorem establishes the reality of the quantum state provided you accept:
  - The ontological models framework
  - The PIP
- I think it is more likely that the ontological models framework is wrong than the PIP.
- There are ψ-ontology theorems with different assumptions. I think these are less plausible[4].
- You can also prove slightly weaker results with a generalization of the PIP[5].
- Next time, we will look at what you can prove without the PIP and how this is related to other no-go theorems.

[4] ML, Quanta 3:67–155 (2014).
[5] S. Mansfield, Phys. Rev. A., 94:042124 (2016).