No Go Theorems 2: Oh No You Didn't!

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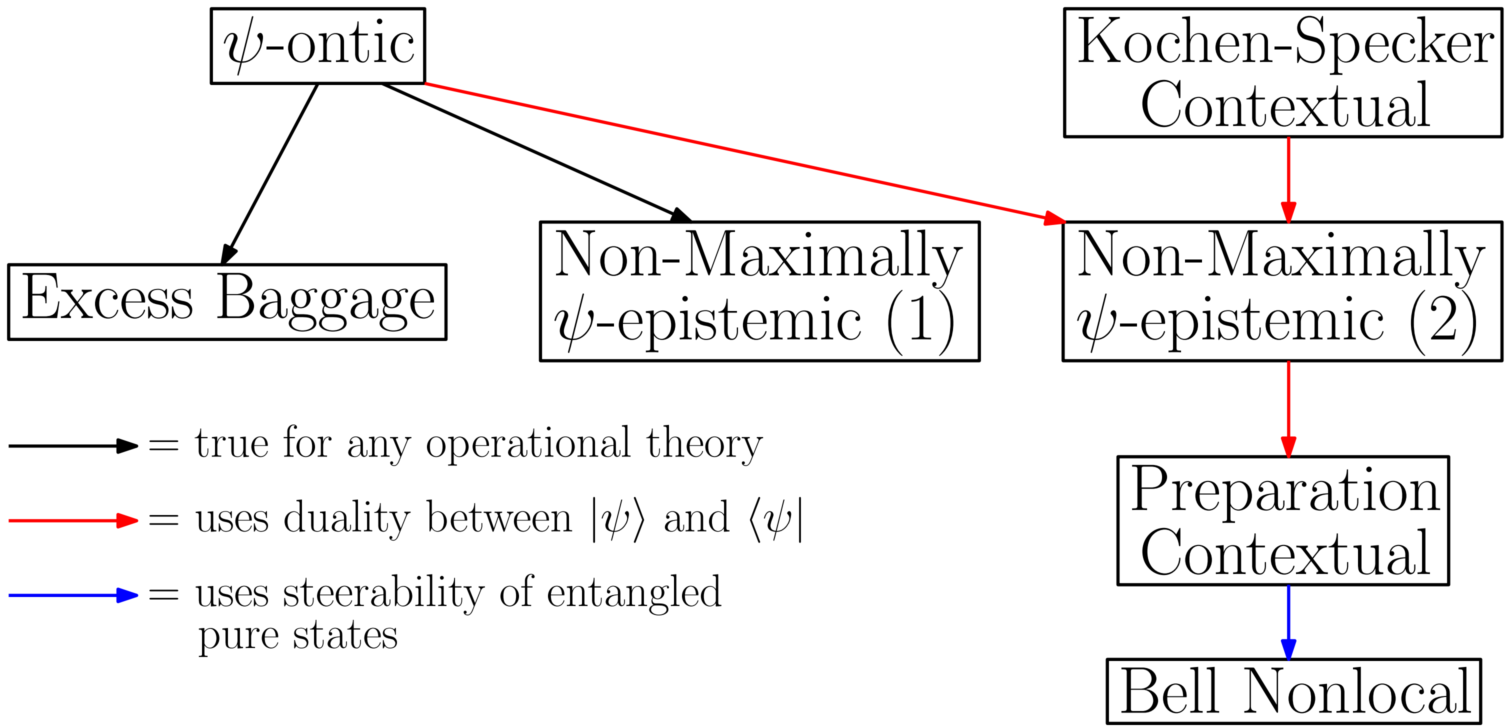
Introduction

Heirarchy of Properties of Ontological Models



Ontological Excess Baggage

How much information is stored in a qubit?

- By the Holevo bound, we can only reliably store 1 bit of classical information in a qubit.
- There are an infinite number of pure quantum states, but there are an infinite number of 1-bit classical probability distributions as well,
  - so in a ψ-epistemic model this is not evidence that there is an infinite amount of information in a qubit.
- Can we construct an ontological model for a qubit with only a finite number of ontic states?
  - For a ψ-ontic model, the answer is no, but proving this requires additional assumptions.
  - Lucien Hardy showed the answer is no in general.
- Since then, Montina has shown
  - Λ must have the cardinality of the continuum.
  - Even an approximate model must have $|\Lambda| = O(e^d)$, where d is Hilbert space dimension.

See D. Jennings and ML, Contemp. Phys. 57:60–82 (2015) for references to original work.

A Useful Lemma

- Today, we will mainly consider preparations of pure states and measurements in orthonormal bases.
- We use naughty notation for preparations Pr$(\lambda|\psi)$, but not for measurements.
- **Lemma**: Consider a preparation of $|\psi\rangle$ and let M be a measurement in an orthonormal basis that includes $|\psi\rangle$. Let,

$$\Lambda_\psi = \{\lambda \in \Lambda | \text{Pr}(\lambda|\psi) > 0\}, \quad \Gamma_\psi^M = \{\lambda \in \Lambda | \text{Pr}(\psi|M,\lambda) = 1\}.$$

  Then $\Lambda_\psi \subseteq \Gamma_\psi^M$ (up to measure-zero sets).

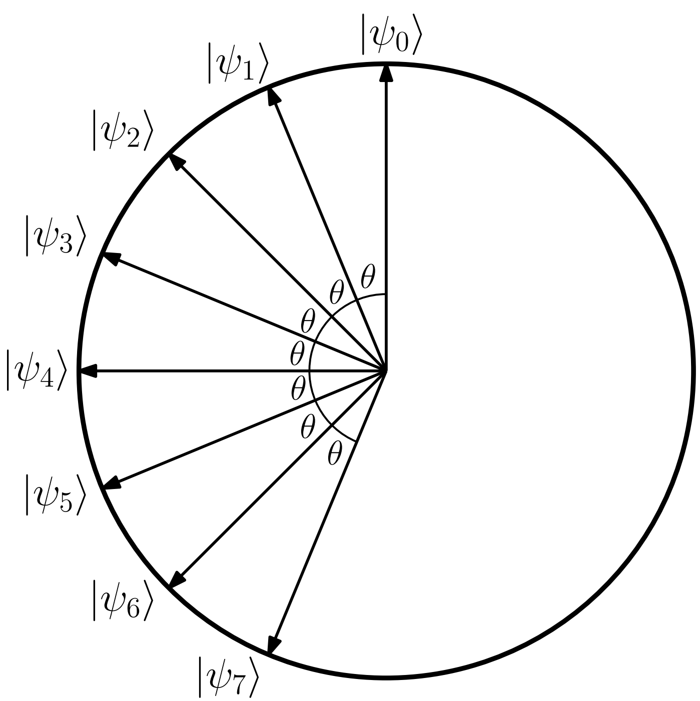
- Proof:

$$1 = |\langle\psi|\psi\rangle|^2 = \int_\Lambda \text{Pr}(\psi|M,\lambda)\text{Pr}(\lambda|\psi)\text{d}\lambda$$
$$= \int_{\Lambda_\psi} \text{Pr}(\psi|M,\lambda)\text{Pr}(\lambda|\psi)\text{d}\lambda.$$

  However, since $\int_{\Lambda_\psi} \text{Pr}(\lambda|\psi)\text{d}\lambda = 1$ and Pr$(\psi|M,\lambda) \leq 1$, Pr$(\psi|M,\lambda)$ must equal 1 almost everywhere on Λ_ψ.

- **Theorem**: Any ontological model that can reproduce the predictions for orthonormal basis measurements on pure states in any Hilbert space dimension must have $|\Lambda| = \infty$.

- Proof:
  - Assume that $|\Lambda| = N$ for some finite N.
  - Consider a 2-dimensional subspace spanned by $|0\rangle$ and $|1\rangle$ and the M states



$$|\psi_j\rangle = \cos\left(\frac{j\pi}{2M}\right)|0\rangle + \sin\left(\frac{j\pi}{2M}\right), \quad j = 0, 1, \ldots, M-1$$

$|\langle\psi_k|\psi_j\rangle|^2 < 1$ for all $j \neq k$

- Consider preparing the system in the state $|\psi_j\rangle$ and measuring it in a basis that includes $|\psi_k\rangle$ for $k \neq j$.
- Then,

$$\sum_{\lambda \in \Lambda} \text{Pr}(\psi_k|\lambda)\text{Pr}(\lambda|\psi_j) < 1.$$

- Hence, there must exist a $\lambda \in \Lambda_{\psi_j}$ such that $\text{Pr}(\psi_k|\lambda) < 1$, otherwise the sum would be 1.
- Since $\text{Pr}(\psi_k|\lambda) = 1$ everywhere on Λ_{ψ_k}, Λ_{ψ_j} and Λ_{ψ_k} must be distinct subsets of Λ.
- This applies to every pair, so there must be at least M distinct subsets of Λ.
- The number of distinct subsets of Λ is 2^N, so

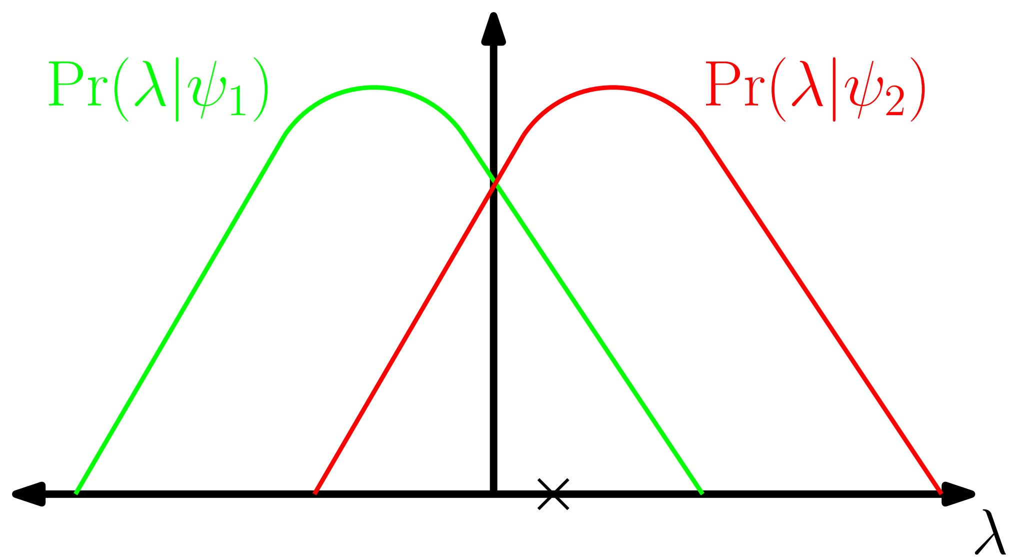
$$2^N \geq M \qquad \text{or} \qquad N \geq \log_2 M.$$

- Since we can choose M as large as we like, N must be larger than any finite integer. Hence, $N = \infty$.

Maximally ψ-epistemic models

The ψ-epistemic explanation of indistinguishability

- In a ψ-epistemic model, the epistemic states corresponding to nonorthogonal pure states can overlap, in which case they cannot be distinguished with certainty because sometimes a λ in the overlap region is prepared.

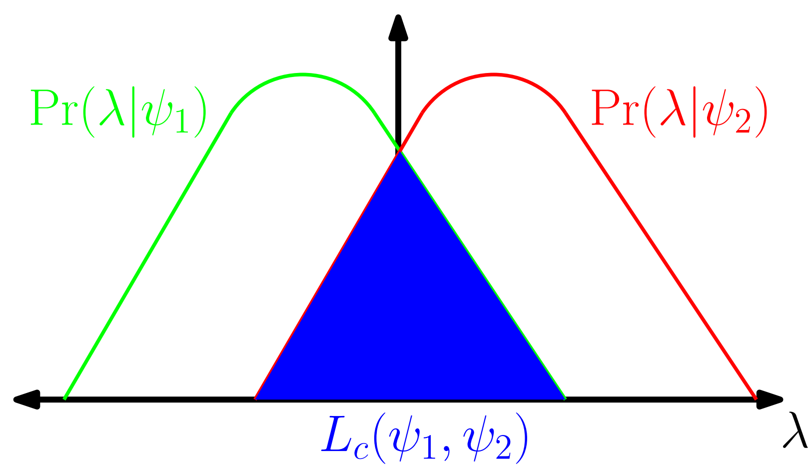


- In order for this to work as an explanation, the amount of overlap needs to be comparable to the degree of indistinguishability.

Classical Symmetric Overlap

- *Classical symmetric overlap*:

$$L_c(\psi_1, \psi_2) = \int_\Lambda \left[\min\{\text{Pr}(\lambda|\psi_1), \text{Pr}(\lambda|\psi_2)\}\right] d\lambda$$



- Optimal success probability of distinguishing $|\psi_1\rangle$ and $|\psi_2\rangle$ if you know λ:

$$p_c(\psi_1, \psi_2) = \frac{1}{2}\left(2 - L_c(\psi_1, \psi_2)\right)$$

Quantum Symmetric Overlap

- *Classical symmetric overlap*:

$$L_c(\psi_1, \psi_2) = \int_\Lambda [\min\{\Pr(\lambda|\psi_1), \Pr(\lambda|\psi_2)\}] \, d\lambda$$

- *Quantum symmetric overlap*:

$$\begin{aligned} L_q(\psi_1, \psi_2) &= \inf_{0 \leq E \leq I} [\langle\psi_1| E |\psi_1\rangle + \langle\psi_2| (I - E) |\psi_2\rangle] \\ &= 1 - \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2} \end{aligned}$$

- Optimal success probability of distinguishing $|\psi_1\rangle$ and $|\psi_2\rangle$ based on a quantum measurement:

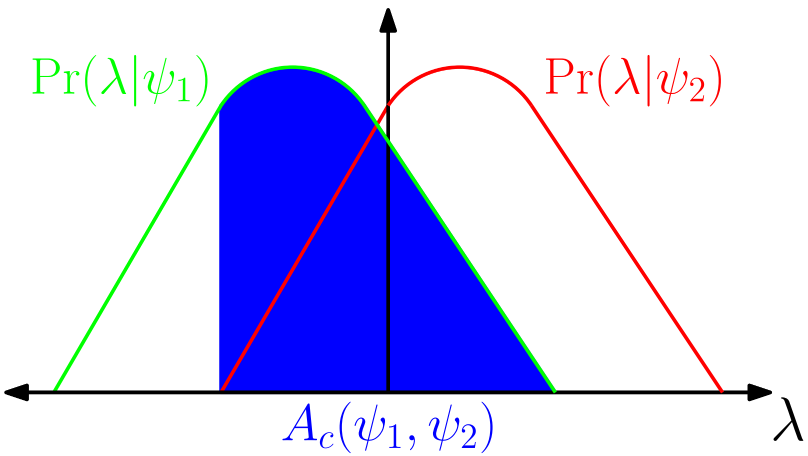
$$p_q(\psi_1, \psi_2) = \frac{1}{2}\left(2 - L_q(\psi_1, \psi_2)\right)$$

- A model is *maximally ψ-epistemic (1)* if $L_c(\psi_1, \psi_2) = L_q(\psi_1, \psi_2)$ for all $|\psi_1\rangle$, $|\psi_2\rangle$.

Classical Ayymmetric Overlap

- *Classical asymmetric overlap*:

$$A_c(\psi_1, \psi_2) = \int_{\Lambda_{\psi_2}} \mathrm{Pr}(\lambda|\psi_1)\mathrm{d}\lambda$$



- $A_c(\psi_1, \psi_2)$ is the amount of the quantum probability of obtaining outcome $|\psi_2\rangle$ when measuring a system prepared in state $|\psi_1\rangle$ that is accounted for by the region of overlap between $\mathrm{Pr}(\lambda|\psi_1)$ and $\mathrm{Pr}(\lambda|\psi_2)$.

Quantum Asymmetric Overlap

- *Classical asymmetric overlap*:

$$A_c(\psi_1, \psi_2) = \int_{\Lambda_{\psi_1}} \Pr(\lambda|\psi_2)\mathrm{d}\lambda$$

- *Quantum asymmetric overlap*:

$$A_q(\psi_1, \psi_2) = |\langle\psi_1|\psi_2\rangle|^2$$

- A model is *maximally ψ-epistemic (2)* if $A_c(\psi_1, \psi_2) = A_q(\psi_1, \psi_2)$ for all $|\psi_1\rangle$, $|\psi_2\rangle$.

- $L_c(\psi_1, \psi_2) \leq A_c(\psi_1, \psi_2)$
- Proof:

$$\begin{aligned} L_c(\psi_1, \psi_2) &= \int_\Lambda \left[\min\{\mathrm{Pr}(\lambda|\psi_1), \mathrm{Pr}(\lambda|\psi_2)\}\right] \mathrm{d}\lambda \\ &= \int_{\Lambda_{\psi_2}} \left[\min\{\mathrm{Pr}(\lambda|\psi_1), \mathrm{Pr}(\lambda|\psi_2)\}\right] \mathrm{d}\lambda \\ &\leq \int_{\Lambda_{\psi_2}} \mathrm{Pr}(\lambda|\psi_1) \mathrm{d}\lambda \\ &= A_c(\psi_1, \psi_2) \end{aligned}$$

- For the remainder of this section, consider an ontological model for measurements in a *finite* set $\mathcal{M}$ of orthonormal bases.
- We previously defined the set:

$$\Gamma_\psi^M = \{\lambda|\text{Pr}(\psi|\lambda, M) = 1\}.$$

- This is the set of ontic states that always return the outcome $|\psi\rangle$ when measurement M is made.
- But $|\psi\rangle$ may appear in more than one orthonormal basis, so we can define:

$$\Gamma_\psi = \cap_{\{M\in\mathcal{M}||\psi\rangle\in M\}}\Gamma_\psi^M.$$

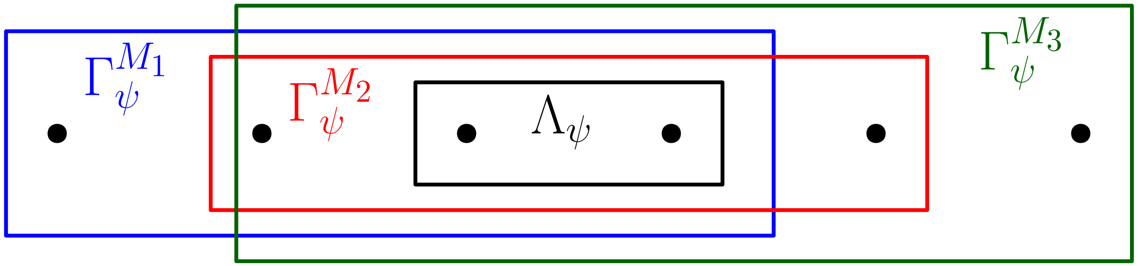
- This is the set of states that always returns the outcome $|\psi\rangle$ regardless of which basis that contains it is measured, i.e. the noncontextual set for $|\psi\rangle$.
- Clearly, in a Kochen-Specker noncontextual model,

$$|\langle\psi_2|\psi_1\rangle|^2 = \int_{\Gamma_{\psi_2}} \text{Pr}(\lambda|\psi_1)\text{d}\lambda.$$

- The converse is also true (up to the removal of measure-zero sets of contextual ontic states).

Maximally ψ-epistemic models (2) are noncontextual

- We proved previously that $\Lambda_\psi \subseteq \Gamma_\psi^M$ (up to sets of measure zero).
- In fact, the stronger result $\Lambda_\psi \subseteq \Gamma_\psi$ also holds.



- Technically, Γ_ψ^M is a measure-one set according to $\text{Pr}(\lambda|\psi)$ and the intersection of a *finite number* of measure-one sets is also measure one.
- Now, in general, we must have

$$A_c(\psi_1, \psi_2) = \int_{\Lambda_{\psi_2}} \text{Pr}(\lambda|\psi_1) d\lambda \leq \int_{\Gamma_{\psi_2}} \text{Pr}(\lambda|\psi_1) d\lambda$$

$$\leq \int_\Lambda \text{Pr}(\psi_2|\lambda, M)\text{Pr}(\lambda|\psi_1) d\lambda = |\langle\psi_2|\psi_1\rangle|^2 .$$

- So, if the model is maximally ψ-epistemic (2) then it is also Kochen-Specker noncontextual (up to measure zero sets).

Overlap Bounds

- Can we quantify the degree to which a model fails to be maximally ψ-epistemic?
- We will do this with symmetric overlaps ((1) definition) because they are experimentally robust.
- Take a finite set $\mathcal{M}$ of orthonormal bases and consider a subset V of the states that occur in $\mathcal{M}$.
- Choose another state $|\psi\rangle$ to compare them with.
- We can compute

$$\bar{L}_q(\psi) = \frac{1}{|V|}\sum_{|\phi\rangle\in V} L_q(\psi,\phi) = \frac{1}{|V|}\sum_{|\phi\rangle\in V}\left(1-\sqrt{1-|\langle\phi|\psi\rangle|^2}\right)$$

.

- We want to upper bound $\bar{L}_c = \frac{1}{|V|}\sum_{|\phi\rangle\in V} L_c(\psi,\phi)$.
- This will give us a lower bound on the average overlap deficit

$$\Delta\bar{L}(\psi) = \bar{L}_q(\psi) - \bar{L}_c(\psi)$$

.

- If $\Delta\bar{L}(\psi)$ is close to 1 then almost all of the states $|\phi\rangle \in V$ are close to being ontologically distinct from $|\psi\rangle$ — strong evidence against the ψ-epistemic explanation of indistinguishability.
- How do we bound $\bar{L}_c(\psi)$?
  - Using Kochen-Specker noncontextuality inequalities.
- We can use

$$L_c(\psi, \phi) \leq A_c(\psi, \phi) \leq \int_{\Gamma_\phi} \text{Pr}(\lambda|\psi)\mathrm{d}\lambda$$

- to obtain

$$\bar{L}_c(\psi) \leq \frac{1}{|V|} \sum_{|\phi\rangle \in V} \int_{\Gamma_\phi} \text{Pr}(\lambda|\psi)\mathrm{d}\lambda.$$

- The RHS is bounded by the maximum probability that can be assigned to the $|\phi\rangle$'s in a KS noncontextual model, i.e. a noncontextuality inequality.

	Dimension	$\|V\|$	$\bar{L}_c(\psi)$	$\bar{L}_q(\psi)$
Barrett et. al.[1]	Prime power $d \geq 4$	d^2	$1/d^2$	$1-\sqrt{1-1/d}$
Leifer[2]	$d \geq 3$	2^{d-1}	$1/2^{d-1}$	$1-\sqrt{1-1/d}$
Branciard[3]	$d \geq 4$	$n \geq 2$	$1/n$	$1-\sqrt{1-\frac{1}{4}n^{-1/(d-2)}}$
Amaral et. al.[4]	$d \geq n_j$	$n_j \geq ?$	$n_j^{\delta-1}$	$1-\sqrt{\frac{1}{2}+\epsilon}$

[1] J. Barrrett et. al., Phys. Rev. Lett. 112, 250403 (2014)
[2] ML, Phys. Rev. Lett. 112, 160404 (2014)
[3] C. Branciard, Phys. Rev. Lett. 113, 020409 (2014)
[4] B. Amaral et. al., Phys. Rev. A 92, 062125 (2015)

Optimizing for Overlap deficit

	Optimal dimension	Optimal $\lvert V \rvert$	$\Delta \bar{L}$
Barrett et. al.	4	16	0.0715
Leifer	7	64	0.0586
Branciard	4	$n \to \infty$	0.134
Amaral et. al.	$d \to \infty$	$n_j \to \infty$	0.293

Experiment

- Ringbauer et. al.[5] experiment (based on Branciard's construction) obtained:

$$\Delta \bar{L} \geq 0.047 \pm 0.010$$

- What should we think about such small numbers?
- In any ontological model there are two mechanisms for explaining the indistinguishability of quantum states:
  - The ψ-epistemic explanation: Pr$(\lambda|\psi_1)$ and Pr$(\lambda|\psi_2)$ overlap.
  - The response functions Pr$(\psi|\lambda, M)$ do not reveal full information about λ.
- Although we expect overlap to play an important role in ψ-epistemic model, there is no good reason why the second explanation should not play a role too.
- Therefore, $\Delta \bar{L}$ needs to be close to 1 in order to have strong evidence against ψ-epistemic models.

[5] M. Ringbauer et. al. Nature Physics 11, 249–254 (2015).

Preparation Contextuality

- We will show that non-maximally ψ-epistemic (2) models must be preparation contextual (and hence Kochen-Specker contextuality implies preparation contextuality).

- Reminder: Two preparations, P_1 and P_2 are *operationally equivalent* if, for all (M, k),

$$\text{Prob}(k|P_1, M) = \text{Prob}(k|P_2, M).$$

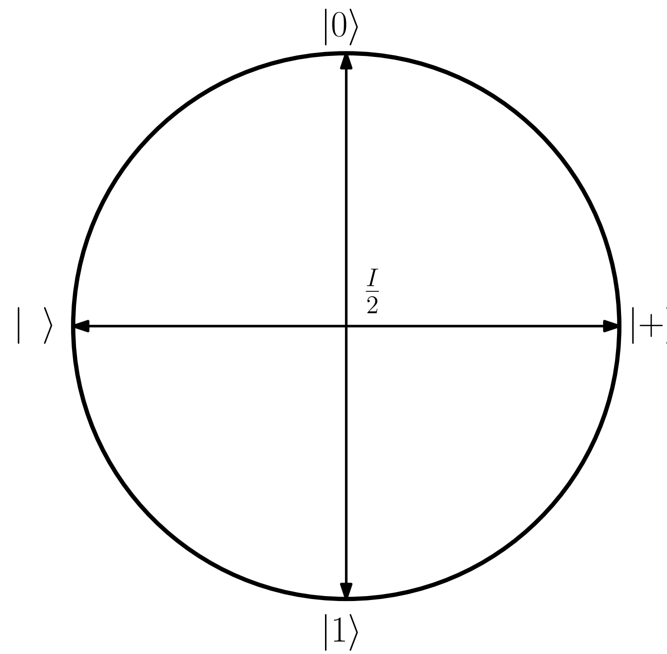
- In quantum theory, preparations that are represented by the same density operator are operationally equivalent.

- An ontological model is *preparation noncontextual* if, whenever P_1 and P_2 are operationally equivalent, then

$$\text{Pr}(\lambda|P_1) = \text{Pr}(\lambda|P_2).$$

Warm up: ψ-ontic models are preparation contextual

- **Theorem**: ψ-ontic models are preparation contextual.
- Proof: Consider the four states: $|0\rangle$, $|1\rangle$, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.



- Since $L_c(\psi_1, \psi_2) = 0$ for every pair of states, Λ_0, Λ_1, Λ_+ and Λ_- are disjoint (up to measure-zero sets).

Warm up: ψ-ontic models are preparation contextual

- Because the maximally mixed state is a $50/50$ mixture of $|0\rangle$ and $|1\rangle$, and also a $50/50$ mixture of $|+\rangle$ and $|-\rangle$, a preparation contextual model must have

$$\frac{1}{2}\text{Pr}(\lambda|0) + \frac{1}{2}\text{Pr}(\lambda|1) = \frac{1}{2}\text{Pr}(\lambda|+) + \frac{1}{2}\text{Pr}(\lambda|-).$$

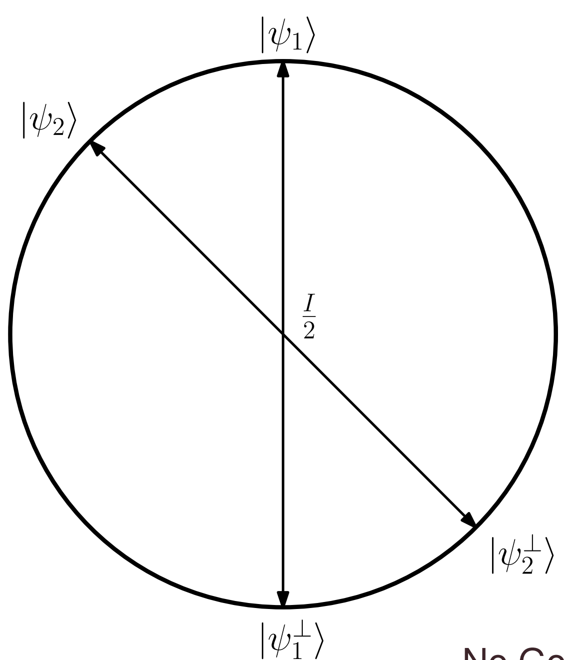
- But for (almost) all $\lambda \in \Lambda_0 \cup \Lambda_1 \cup \Lambda_+ \cup \Lambda_-$, only one of the terms is nonzero.

Non-maximally ψ-epistemic (2) models are preparation contextual

- **Theorem**: Non-maximally ψ-epistemic (2) models are preparation contextual.
- Proof: If a model is non-maximally ψ-epistemic (2) then there exists a pair of states, $|\psi_1\rangle$ and $|\psi_2\rangle$, such that

$$\int_{\Lambda_{\psi_2}} \text{Pr}(\lambda|\psi_1)\mathrm{d}\lambda < |\langle\psi_2|\psi_1\rangle|^2 .$$

- Consider the two-dimensional subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$. Let $|\psi_1^\perp\rangle$ and $|\psi_2^\perp\rangle$ be states in this subspace such that $\left|\left\langle\psi_1^\perp\middle|\psi_1\right\rangle\right|^2 = 0$ and $\left|\left\langle\psi_2^\perp\middle|\psi_2\right\rangle\right|^2 = 0$.



Non-maximally ψ-epistemic (2) models are preparation contextual

- In order to reproduce the quantum predictions, there must be a set $\Omega \subseteq \Lambda \backslash \Lambda_{\psi_2}$ such that $\text{Pr}(\psi_2|\lambda) > 0$ everywhere in Ω and $\int_\Omega \text{Pr}(\lambda|\psi_1)\mathrm{d}\lambda > 0$.

- It is also the case that $\int_\Omega \text{Pr}(\lambda|\psi_2)\mathrm{d}\lambda = 0$ because Ω is disjoint from Λ_{ψ_2}.

- Now, we must also have

$$\int_\Omega \text{Pr}(\psi_2|\lambda)\text{Pr}(\lambda|\psi_2^\perp)\mathrm{d}\lambda \leq |\langle\psi_2|\psi_1\rangle|^2 = 0,$$

  so $\int_\Omega \text{Pr}(\lambda|\psi_2^\perp)\mathrm{d}\lambda = 0$ because $\text{Pr}(\psi_2|\lambda) > 0$ everywhere in Ω.

- The maximally mixed state can be prepared as a $50/50$ mixture of $|\psi_1\rangle$ and $|\psi_1^\perp\rangle$, or as a $50/50$ mixture of $|\psi_2\rangle$ and $|\psi_2^\perp\rangle$.

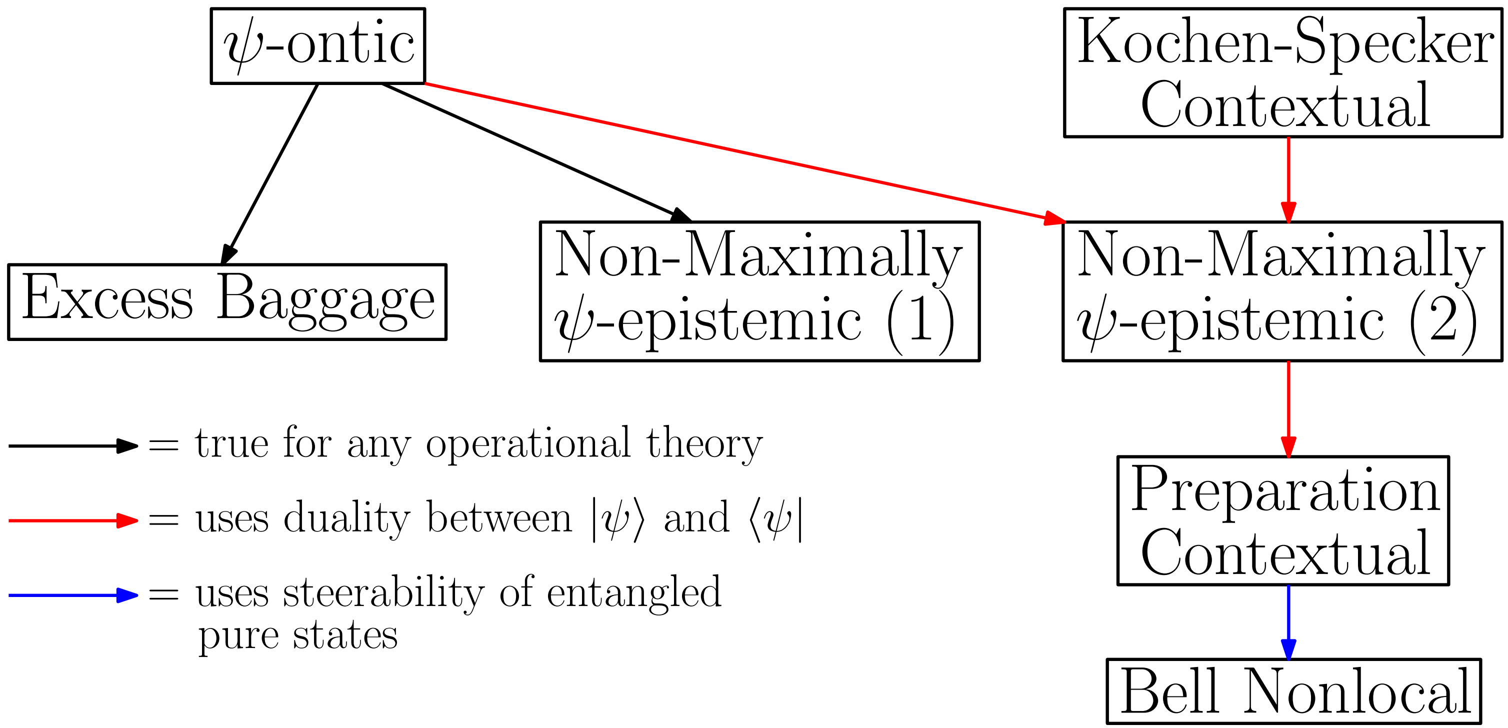
- So, in a preparation noncontextual model, we must have:

$$\frac{1}{2}\text{Pr}(\lambda|\psi_1) + \frac{1}{2}\text{Pr}(\lambda|\psi_1^\perp) = \frac{1}{2}\text{Pr}(\lambda|\psi_2) + \frac{1}{2}\text{Pr}(\lambda|\psi_2^\perp).$$

- Integrate both sides over Ω. The LHS is > 0 but the RHS $= 0$. Hence, we cannot have a preparation noncontextual model.

Conclusions

Heirarchy of Properties of Ontological Models



Conclusions

- Proving that models of quantum theory must be ψ-ontic would imply many existing no-go theorems, but we cannot do so without the PIP.
- Kochen-Specker contextuality has most of the same implications, but it does not imply excess baggage.
- It is still possible that models of infinite dimensional Hilbert spaces, or finite dimensional Hilbert spaces with POVMs must be ψ-ontic.
- Existing overlap bounds are fairly weak. It is possible that other contextuality inequalities and/or methods not based on contextuality could give better bounds.
- What next for ψ-epistemicists?
  - Adopt a Copenhagenish interpretation.
  - Adopt a more exotic ontology: e.g. retrocausality, ironic many-worlds, ?