Causality in Quantum Theory

Katja Ried, Univ. Innsbruck

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Itinerary

1. Causality (classical version)
   - concepts and formalism
   - features and phenomena
2. Quantum indeterminism
3. Retrocausality and causal loops
4. Nonlocality
   - in quantum field theory
   - Bell inequality violations
5. Quantum causal models
   - concepts and formalism
6. Causal structure (quantum version)
   - non-classical causal relations
   - experiments
Causality
(classical version)
On what grounds can one say that A causes B?

Some more specific questions:

Are the answers different talking about events or variables?

Can we always identify causes, and are they unique?

Under which circumstances can A not causally influence B?

Meta-question: What does this definition of causality accomplish?
(philosophical insights, operational predictions...)
Some possible answers

Aristotle's four causes:
(1) material cause: constituent matter, (2) formal cause: shape, arrangement, (3) efficient or moving cause: agents (4) final cause: purpose. [Falcon]

„We may define a cause to be an object, followed by another, and where all the objects similar to the first are followed by objects similar to the second.“ [Hume]

„If an improbable coincidence has occurred, there must exist a common cause.“ [Reichenbach]

In quantum field theory, „for our theory to be causal, we must require that all spacelike separated operators commute“. [Tong]

An event A is statistically independent of its non-descendants given its causal parents. [Pearl]
Causal Models: conceptual and mathematical framework

Origin: statistics – sociology, epidemiology, econometrics...

System modelled: relations among a set of coarse-grained random variables

Structure
pattern of influences

directed acyclic graph

Parameters
description of influences

\[ P(B|AG) \]
\[ P(A|G) \]
\[ P(G) \]

conditional probabilities
Causal Models: conceptual and mathematical framework

Origin: statistics – sociology, epidemiology, econometrics...

System modelled: relations among a set of coarse-grained random variables

Mathematical toolbox:
- conditional probabilities: \( P(Y|X) \geq 0 \quad \sum_y P(Y|X) = 1 \quad \forall x \)
- belief propagation: \( P(Y) = \sum_x P(Y|X=x) P(X=x) \)
Causal Models: features and phenomena

Causation defined in terms of **interventions**
A has a causal effect on B if we can change the value of B by manipulating A (while leaving all else unchanged).

- practical significance: control
Causal Models: features and phenomena

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- assumption of free will (or at least sufficiently strong randomness)
Causal Models: features and phenomena

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- practical significance: control
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\[
\frac{P(B | AG)}{P(A | G)} \quad \frac{P(A | G)}{P(G)}
\]

effect on conditionals: intervention overrides incoming causal influences

structural effect:
surgery on DAG
Causal Models: features and phenomena

Causation defined in terms of **interventions**
A has a causal effect on B if we can change the value of B by manipulating A (while leaving all else unchanged).

- practical significance: control
- assumption of free will (or at least sufficiently strong randomness)

\[
P\left( \frac{B}{AG} \right) \quad \frac{P(A|G)}{P(G)}
\]

effect on conditionals: intervention overrides incoming causal influences

Local intervention leaves all other causal mechanisms unaffected: 
**autonomy of causal mechanisms**

structural effect: surgery on DAG
Causal Models: features and phenomena

simple picture of interventions:
incoming causal influences
are eliminated

\[
\frac{P(B \mid AG)}{P(A \mid G)}
\]

\[
\frac{P(A \mid G)}{P(G)}
\]
Causal Models: features and phenomena

simple picture of interventions:
incoming causal influences
are eliminated

\[
P(B | AG) \quad \frac{P(A | G)}{P(G)}
\]

more informative perspective: **splitting** of variables

randomization

\[
P(B | A_2 G) \\
P(A_2 | A_1, I) \\
P(A_1 | G) \\
P(G)
\]
Causal Models: features and phenomena

simple picture of interventions:
incoming causal influences
are eliminated

\[
P(B | AG) = \frac{P(A | G)}{P(G)}
\]

more informative perspective:
\textbf{splitting} of variables

assigned treatment
randomization
intent to treat

unobserved
common cause

recovery
Causal Models: features and phenomena

Interventions can be problematic.

Randomized trial on the health effects of smoking

Solution: causal inference
Causal Models: features and phenomena

**Causal inference**: discovering causal relations without interventions

- ice cream consumption
- drowning incidents
Causal Models: features and phenomena

**Causal inference**: discovering causal relations without interventions
Causal Models: features and phenomena

**Causal inference**: discovering causal relations without interventions

*Conditional independence*: two variables become statistically independent when one conditions on a third

\[ I \perp D \mid T \]

„If an improbable coincidence has occurred, there must exist a common cause.“ [Reichenbach]
Causal Models: features and phenomena

Conditional independences can arise from various causal structures.

Structures that lead to $X \perp Y \mid Z$:

(a) chain

(b) fork

(c) collider
Causal Models: features and phenomena

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Structures that lead to $X \perp Y \mid Z$:

- (a) chain
- (b) fork
- (c) collider

d-separation
(property of causal structure)

$\Downarrow \uparrow$?

conditional independence
(property of probability distribution)
Causal Models: features and phenomena

Conditional independences can arise from various causal structures.

Structures that lead to $X \perp Y \mid Z$:

(a) chain

X \xrightarrow{not a good day} Z \xrightarrow{not a good day} Y

faulty aircon

兴致勃勃

(b) fork

X \xrightarrow{left sock} Y \xrightarrow{choice this morning} Z

left sock

right sock

选择这个早晨

(c) collider

W \xrightarrow{academic job} Z \xrightarrow{sock colour preference} Y

W

academic job

Z

sock colour preference

Choice

Conditional independences reveal features of causal structure if we exclude fine-tuning.

x = \alpha z + u_x

y = \beta z + \gamma x + u_y

\begin{align*}
\end{align*}
Causal Models: features and phenomena

Conditional independences can arise from various causal structures.

Structures that lead to $X \perp Y \mid Z$:

(a) chain

(b) fork

(c) collider

d-separation

(property of causal structure)

conditional independence

(property of probability distribution)

Conditional independences reveal features of causal structure if we exclude fine-tuning.

$x = \alpha z + u_x$

$y = \beta z + \gamma x + u_y$

$\gamma = 0 \Rightarrow Y \perp X \mid Z$

$\gamma = -\beta / \alpha \Rightarrow Y \perp (X, Z)$
Causal Models: features and phenomena

Functional causal models:
alternative description with deterministic relations

\[ B = f_B(A, G, u_B) \]
\[ A = f_A(G, u_A) \]
\[ G = f_G(u_G) \]

coarse-graining
fine-graining

\[ P(B|AG) \]
\[ P(A|G) \]
\[ P(G) \]

autonomy ↔ independence of noise sources
Causal Models: features and phenomena

Example: using the independence of mechanisms for causal inference

Data: joint probability distribution over two variables, $x = \{-1, +1\}$ and $y$
Causal Models: features and phenomena

Example: using the independence of mechanisms for causal inference

Data: joint probability distribution over two variables, \( x = \{-1, +1\} \) and \( y \)

- **Explanation 1: \( x \rightarrow y \)**

  \[
P(x) = \frac{1}{2} \quad \forall \ x
  
  P(y|x) = \alpha \exp\left(\frac{-(y-x)^2}{\mu}\right)
  \]

- **Explanation 2: \( y \rightarrow x \)**

[Example and figure from Janzing and Schölkopf, arXiv:0804.3678]
Causal Models: features and phenomena

**Time** in causal models

„We may define a cause to be an object, *followed by another*, and where all the objects similar to the first are followed by objects similar to the second.“ [Hume]
Causal Models: features and phenomena

**Time** in causal models

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The class of continuous timelike curves determines the topology of spacetime. [Malament, also Hawking et al]
Time in causal models

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Causal structure is acyclic.
Causal Models: features and phenomena

Using causal information: **inference**

Example 1: Given a causal model, derive joint and marginal probability distributions.

Conditional probabilities: \( P(X|Y) \geq 0 \) \( \sum_{x} P(X|Y) = 1 \forall y \)
Causal Models: features and phenomena

Using causal information: **inference**

Example 1: Given a causal model, derive joint and marginal probability distributions.

- **joint prob. distrib.** $P(IDT)$

<table>
<thead>
<tr>
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<th>I=0 D=0</th>
<th>I=0 D=1</th>
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- **marginal prob. distrib.** $P(ID)$

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<td>D=1</td>
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- **Conditional probabilities:** $P(I|T)$, $P(D|T)$

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<tr>
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<td>T=1</td>
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- ** marginal prob. distrib.** $P(ID)$

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- **joint prob. distrib.** $P(IDT)$

$P(T) = \{.6, .4\}$
Causal Models: features and phenomena

Using causal information: **inference**

Example 2: Quantum foundations in Zurich

\[
P(Z=1|Q) = \begin{cases} 
10^{-4} & (Q=0) \\ 
10^{-1} & (Q=1) 
\end{cases}
\]

\[
P(Q|Z=1) = ?
\]
Causal Models: features and phenomena

Using causal information: *inference*

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Causal Models: features and phenomena

Using causal information: inference

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\]

retrodiction: inference about the causal past

Bayesian inversion:

\[
P(Z, Q) = P(Z|Q)P(Q)
\]

\[
\rightarrow P(Q|Z) = \frac{P(Z, Q)}{P(Z)}
\]

| Z  | P(Q|Z)   | Q=0   | Q=1   |
|----|---------|-------|-------|
| Z=0| 1-10^{-7}| 10^{-7}|
| Z=1| 1-10^{-4}| 10^{-4}|

\[
P(Q|Z) = \begin{cases} 
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Causal Models: features and phenomena

Using causal information: **inference**

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\begin{array}{c|c|c}
\text{P(Q|Z)} & \text{Q=0} & \text{Q=1} \\
\hline
Z=0 & 1-10^{-7} & 10^{-7} \\
Z=1 & 10^{-4} & 10^{-4} \\
\end{array}
\]

**causal vs inferential conditionals**

- same mathematical form
- different epistemological significance
Causal Models: features and phenomena

Using causal information: **inference**

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|-------|-------|------|------|
| Z=0   | 1-10^{-7} | 10^{-7} |
| Z=1   | 1-10^{-4}  | 10^{-4}  |

**causal vs inferential conditionals**

- same mathematical form
- different epistemological significance

Example 3: Correlation with coffee

\[
P(C=1|Q) = \begin{cases} .5 & (Q=0) \\ .8 & (Q=1) \end{cases}
\]

\[
P(C|Z=1) = \ ?
\]

inference via a common cause
Causal Models: features and phenomena

Using causal information: **inference**

Example 2: Quantum foundations in Zurich

\[ P(Z = 1|Q) = \begin{cases} 10^{-4} & (Q = 0) \\ 10^{-1} & (Q = 1) \end{cases} \]

\[ P(Q|Z) = ? \quad P(Q = 1) = 10^{-7} \]

**retrodiction**: inference about the causal past

Bayesian inversion:

\[ P(Z, Q) = P(Z|Q)P(Q) \]

\[ \rightarrow P(Q|Z) = \frac{P(Z, Q)}{P(Z)} \]

| P(Q|Z) | Q=0   | Q=1   |
|--------|-------|-------|
| Z=0    | 1-10^{-7} | 10^{-7} |
| Z=1    | 1-10^{-4}  | 10^{-4}  |

**causal vs inferential conditionals**

- same mathematical form
- different epistemological significance

Example 3: Correlation with coffee

\[ P(C = 1|Q) = \begin{cases} .5 & (Q = 0) \\ .8 & (Q = 1) \end{cases} \]

\[ P(C|Z = 1) = ? \]

inference via a common cause

\[ P(C|Z) = \sum_q P(C|Q = q)P(Q = q|Z) \]
\[ = \sum_q P(C, Q = q|Z) \]

Note

- two-step process:
  joint \( P(CQ|Z) \), then marginalize
- same mathematical form as before
Highlights: classical causal models

• definition of causation based on interventions
• formal consequence: splitting of variables
• causation vs inference: mathematically similar but conceptually distinct

Some essential features:
• autonomy of causal mechanisms
  → no fine-tuning: conditional independences reflect features of causal structure
• admit an account in terms of underlying deterministic mechanisms
• causal order: acyclic, aligned with temporal order
Quantum Indeterminism
Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, Institute for Advanced Study, Princeton, New Jersey
(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.
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Bohmian trajectories in a double-slit experiment
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- open question
- hidden variable models are possible – if one gives up other assumptions
- exploiting quantum indeterminism:
Retrocausality and Causal Loops
The delayed-choice quantum eraser

D(A)  BS1(A)  emitter
D(+)  BS2
D(-)  BS1(B)
D(B)  screen

photon 1: quick detection
photon 2: delayed

[Kim et al, PRL 84, 1 (2000)]
The delayed-choice quantum eraser

[Kim et al, PRL 84, 1 (2000)]
The delayed-choice quantum eraser

\[ \frac{1}{\sqrt{2}} \left( |A\rangle + e^{i\phi} |B\rangle \right) \xrightarrow{BS1} \frac{1}{2} \left( (|A\rangle + |A'\rangle) + e^{i\phi} (|B\rangle + |B'\rangle) \right) \]

\[ \xrightarrow{BS2} \frac{1}{2} \left( |A\rangle + e^{i\phi} |B\rangle + \frac{1}{\sqrt{2}} (1 + e^{i\phi}) |+\rangle + \frac{1}{\sqrt{2}} (1 - e^{i\phi}) |-\rangle \right) \]

probability of learning (i) path: \( P(A) + P(B) = \frac{1}{2} \); (ii) phase: \( P(+) + P(-) = \frac{1}{2} \)

One cannot control which information one acquires, only post-select.

[Kim et al, PRL 84, 1 (2000)]
The two-state vector formalism

The two-state vector formalism (TSVF) [1] is a time-symmetric description of the standard quantum mechanics originated in Aharonov, Bergmann and Lebowitz [2]. The TSVF describes a quantum system at a particular time by two quantum states: the usual one, evolving forward in time, defined by the results of a complete measurement at the earlier time, and by the quantum state evolving backward in time, defined by the results of a complete measurement at a later time.

conventional quantum mechanics:

\[ P(m|\psi) = |\hat{\Pi}_m \psi|^2, \quad P(\phi, m|\psi) = |\phi \hat{\Pi}_m |\psi|^2 \]

\[ \Rightarrow P(m|\psi, \phi) = \frac{|\phi \hat{\Pi}_m |\psi|^2}{\sum_m |\phi \hat{\Pi}_m |\psi|^2} \]

TSVF: backward in time forward in time

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TSVF: backward in time forward in time

„We cannot, however, create with certainty a particular backward evolving quantum state, (...) The difference follows from the time asymmetry of the memory arrow of time.“

Closed timelike curves
Closed timelike curves

\[ y \oplus x = y \Rightarrow x = 0 \]

consistency condition:

\[ y \oplus x = y \Rightarrow x = 0 \]
Closed timelike curves

Quantum mechanics avoids „'paradoxical' constraints on the past“.

Consistency conditions:

$$\rho = Tr_2 \left[ U (\rho_i \otimes \rho) U^\dagger \right]$$

$$\rho_o = Tr_1 \left[ U (\rho_i \otimes \rho) U^\dagger \right]$$

[Ralph&Myers]
Closed timelike curves

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Every \( \rho_i \) admits a solution. [Deutsch]
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Every \( \rho_i \) admits a solution. [Deutsch]

„Qubit's view“ assumes infinitely many copies of \( \rho_i \), leading to

- non-linear transformations
- perfect state discrimination and cloning
- inequivalence of probabilistic mixtures, breaking entanglement
- instant computation: Pspace [Aaronson&Watrous]
- ...

[Ralph&Myers]
Nonlocality
Quantum Field Theory

Consider a classical, free, real scalar field $\phi(\vec{x}, t)$.

non-relativistic case: Schrödinger field

\[
i \frac{\partial}{\partial t} \phi = -\frac{\nabla^2}{2m} \phi \quad \Rightarrow \quad \phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} a(\vec{k}) e^{i\vec{k} \cdot \vec{x}} e^{-i\omega_0 t}, \quad \omega_0 = \frac{|\vec{k}|^2}{2m}
\]
Quantum Field Theory

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**non-relativistic case:** Schrödinger field

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$\vec{k}$ indexes independent modes (harmonic oscillators):

$$\mathcal{H} = \frac{1}{2m} \int d^3 x \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi = \int \frac{d^3 k}{(2\pi)^3} a^*(\vec{k}) a(\vec{k})$$
Quantum Field Theory

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$$\mathcal{H} = \frac{1}{2m} \int d^3 x \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi = \int \frac{d^3 k}{(2\pi)^3} a^*(\vec{k}) a(\vec{k})$$

promote $a(\vec{k})$ to annihilation operators, $\phi(\vec{x}, t)$ to field (annihilation) operator:

$$\hat{\phi}^\dagger(\vec{x}, t) |0\rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 t} \hat{a}^\dagger(\vec{k}) |0\rangle \quad \text{single localized excitation}$$

Note: $\hat{\phi}(\vec{x}, t)$ is an observable.
Quantum Field Theory

Consider a classical, free, real scalar field $\phi(\vec{x}, t)$.

relativistic case: Klein-Gordon field

\[
\frac{\partial^2}{\partial^2 t} \phi = (\nabla^2 - m^2) \phi \quad \Rightarrow \quad k_0^2 = |\vec{k}|^2 + m^2
\]

\[
\phi(\vec{x}, t) = \int \frac{d^3 k}{2k_0 (2\pi)^3} \left[ a(\vec{k}) e^{i\vec{k} \cdot \vec{x}} e^{-i|k_0|t} + a^*(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} e^{+i|k_0|t} \right]
\]

promote $a(\vec{k})$ to annihilation operators, $\phi(\vec{x}, t)$ to field (annihilation) operator:

\[
\hat{\phi}(\vec{x}, t) = \int \frac{d^3 k}{2k_0 (2\pi)^3} \left[ \hat{a}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} e^{-i|k_0|t} + \hat{a}^*(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} e^{+i|k_0|t} \right]
\]

\[
\Rightarrow \quad \begin{cases} 
\text{creates localized excitations: } \hat{\phi}^+(\vec{x}, t)|0\rangle \\
\text{is an observable: } \hat{\phi}^+(\vec{x}, t) = \hat{\phi}(\vec{x}, t)
\end{cases}
\]
Quantum Field Theory

**propagator**: probability amplitude of propagation between \( x \) and \( x' \)

\[
D(x-x') = \langle 0 | \hat{\phi}(x) \hat{\phi}^+(x') | 0 \rangle = \int \frac{d^3k}{2k_0(2\pi)^3} e^{-ik(x-x')} \langle 0 | \hat{a}(\vec{k}) \hat{a}^+(\vec{k}) | 0 \rangle
\]

(compare with the more familiar \( \langle \vec{x} | U(t, t') | \vec{x}' \rangle \))
Quantum Field Theory

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Quantum Field Theory

**propagator**: probability amplitude of propagation between $x$ and $x'$

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(compare with the more familiar $\langle \vec{x} | U(t, t') | \vec{x}' \rangle$)

One can:

- passively observe **two-point correlations**
  between spacelike separations

$$D \approx e^{-imt}$$

$$D \approx e^{-mx}$$
Quantum Field Theory

**propagator**: probability amplitude of propagation between $x$ and $x'$

$$D(x-x') = \langle 0 | \hat{\phi}(x) \hat{\phi}^\dagger(x') | 0 \rangle = \int \frac{d^3 k}{2k_0 (2\pi)^3} e^{-ik(x-x')} \langle 0 | \hat{a}(\vec{k}) \hat{a}^\dagger(\vec{k}) | 0 \rangle$$

(compare with the more familiar $\langle \vec{x} | U(t, t') | \vec{x}' \rangle$)

One can:

- passively observe **two-point correlations** between spacelike separations
- signal by measuring the field observable:

$$\left[ \langle 0 | \hat{\phi}(x') \hat{\phi}(x) \hat{\phi}^\dagger(x') | 0 \rangle \right] - \langle 0 | \hat{\phi}(x') \hat{\phi}(x) | 0 \rangle$$

depends on the **commutator** $[\hat{\phi}(x), \hat{\phi}^\dagger(x')]$
Bell Inequality Violations

outcome $A$

setting $S$

outcome $B$

setting $T$
Postulates

1. **Free choice**: A freely chosen action has no relevant causes.
   (Any cause of an event is in its past.)

2. **Relativistic causality**: The past is the past light-cone.

3. **Common causes**: If two events are correlated and neither is a cause of the other, then they have a common cause that explains the correlation.

4. **Decorrelating explanation**: A common cause \( C \) explains a correlation only if conditioning on \( C \) eliminates the correlation.

Consequences

1. **Agent-causation**: If a relevant event \( A \) is correlated with a freely chosen action, then that action is a cause of \( A \).

2. **Reichenbach**: If two events are correlated, and neither is a cause of the other, then they have a common cause \( C \), such that conditioning on \( C \) eliminates the correlation.

3. **Local causality**: If two space-like separated events \( A \) and \( B \) are correlated, then there is a set of events \( C \) in their common Minkowski past such that conditioning on \( C \) eliminates the correlation.

4. **No superdeterminism**: All events on a space-like hypersurface are uncorrelated with freely chosen actions subsequent to that SLH.

5. **Locality**: The probability of an observable event \( A \) is unchanged by conditioning on a space-like-separated free choice \( b \), even if it is already conditioned on other events not in the future light-cone of \( b \).

6. **Local causality**: If two space-like separated events are correlated, then there is a set of events \( C \) in their common Minkowski past such that conditioning on \( C \) eliminates the correlation.
What can causal inference tell us about Bell experiments?

Inputs: conditional independences
- between settings: $S \perp T$
- no signalling: $A \perp T \mid S$, $B \perp S \mid T$

Some proposed causal structures:

- superluminal influences
- superdeterminism
- retrocausality

[Wood&Spekkens]
What can causal inference tell us about Bell experiments?

Inputs: conditional independences
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Some proposed causal structures:

- superluminal influences
- superdeterminism
- retrocausality

$\Rightarrow$ Classical causal models cannot explain Bell inequality violations because this would require fine-tuning.
Quantum Causal Models

How to describe causal relations between quantum systems?
Ansatz I: via mathematical formalism

the universe and everything
Ansatz I: via mathematical formalism

Probing is described by a quantum instrument:
- map from input to output states
  \[ M^{sr} : \mathcal{L}(\mathcal{H}_i) \rightarrow \mathcal{L}(\mathcal{H}_o) \]
- completely positive
  \[ (M^{sr} \otimes T_B)(\rho_{A,B}) \geq 0 \quad \forall \rho, s, r \]
- sum over results trace-preserving
  \[ Tr[\sum_r M^{sr}(\rho)] = 1 \quad \forall \rho, s \]
Ansatz I: via mathematical formalism

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  \[ \mathcal{M}^{sr} : \mathcal{L}(\mathcal{H}_i) \to \mathcal{L}(\mathcal{H}_o) \]
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- sum over results trace-preserving
  \[ Tr[\sum_r \mathcal{M}^{sr}(\rho)] = 1 \quad \forall \rho, s \]

Example 1: preparing an ensemble \( \{\rho^s\}_s \)
\[ \mathcal{M}^{sr} = \rho^s_o \otimes Tr_i \quad (r \text{ fixed}) \]

Example 2: projective measurement \( \{\Pi^r\}_r \)
\[ \mathcal{M}^{sr}(\rho) = Tr_i(\Pi^r \rho) \quad (s \text{ fixed}) \]
Ansatz I: via mathematical formalism

Probing is described by a *quantum instrument*:
- map from input to output states
  \[ \mathcal{M}_{sr} : \mathcal{L}(\mathcal{H}_i) \rightarrow \mathcal{L}(\mathcal{H}_o) \]
- completely positive
  \[ (\mathcal{M}_{sr} \otimes \mathcal{I}_B)(\rho_{A,B}) \geq 0 \quad \forall \rho, s, r \]
- sum over results trace-preserving
  \[ Tr[\Sigma_r \mathcal{M}_{sr}(\rho)] = 1 \quad \forall \rho, s \]

Equivalent representation: Choi operator

\[ M_{sr} \in \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_o) : \]
\[ \mathcal{M}_{sr}(\rho) = Tr_i[M_{io}^{sr} \rho_i] = \tilde{\rho}_o^{sr} \]

[Choi, Jamiołkowski]
Ansatz I: via mathematical formalism

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  \[ (M^{sr} \otimes \mathcal{I}_B) (\rho_{A,B}) \geq 0 \quad \forall \rho, s, r \]
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Equivalent representation: Choi operator

\[ M^{sr} \in \mathcal{L} (\mathcal{H}_i \otimes \mathcal{H}_o) : \]
\[ M^{sr} (\rho) = Tr_i \left[ M^{sr}_{io} \rho_i \right] = \tilde{\rho}^{sr}_o \]
\[ M^{sr} \geq 0, \quad Tr_o \left[ \sum_r M^{sr}_{io} \right] = I_i \]

[Choi, Jamiołkowski]
Ansatz I: via mathematical formalism

The environment is also described by an operator:

\[ P(r|s) = \text{Tr} \left[ M^{sr}_{io} W_{io} \right] \]

Example: environment prepares a state

\[ W_{io} = \rho_i \otimes I_o \]
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Example: environment prepares a state

\[ W_{io} = \rho_i \otimes I_o \]

Physical constraints: probabilities must be
- non-negative

\[ \text{Tr}\left[ M^{sr} W \right] \geq 0 \quad \forall M \quad \Rightarrow \quad W \geq 0 \]
- normalized

\[ \text{Tr}\left[ \sum_r M^{sr} W \right] = 1 \quad \forall M \]

Counter-example: 'looking for' a particular state

\[ W_{io} = \rho_i \otimes |\psi\rangle \langle \psi|_o \]

\[ \text{Tr}\left[ \sum_r M^{sr} W \right] = 0 \quad \text{if} \quad M^{00} = \frac{1}{d} I_i \otimes |\phi\rangle \langle \phi|_o , \quad |\phi\rangle \perp |\psi\rangle \]
Ansatz I: via mathematical formalism

\[ P(r, q|s, t) = Tr \left[ M_{Ai, Ao}^{sr} \otimes \tilde{M}_{Bi, Bo}^{tq} W_{Ai, Ao, Bi, Bo} \right] \]

Constraints:
\[ Tr \left[ \bar{M}_{Ai, Ao, Bi, Bo}^{sr} W \right] \geq 0 \quad \forall \bar{M} \implies W \geq 0 \]
\[ Tr \left[ \sum_{r, q} M_{Ai, Ao}^{sr} \otimes \tilde{M}_{Bi, Bo}^{tq} W \right] = 1 \quad \forall M, \tilde{M} \]

Various related formalizations:
- quantum combs [Chiribella et al]
- process matrix [Oreshkov et al]
- causal map [Ried et al]
Ansatz II: modification of classical causal models

\[
P(B) = \sum_A P(B | A) P(A)
\]

[Leifer&Spekkens]
Ansatz II: modification of classical causal models

\[ P(G) \]
\[ P(A|G) \]
\[ P(B|AG) \]

belief propagation
\[ P(B) = \sum_A P(B|A)P(A) \]

[Leifer&Spekkens]
Ansatz II: modification of classical causal models

\[ P(G) \]
\[ P(A|G) \]
\[ P(B|AG) \]

**Directed acyclic graph**

**Intervention**

**Hilbert space**

**Density operator**
\[ \rho_G \geq 0, \quad \text{Tr} \rho_G = 1 \]

**Belief propagation**
\[ P(B) = \sum_A P(B|A) P(A) \]

[Leifer&Spekkens]
Ansatz II: modification of classical causal models

\[ P(G) \]
\[ P(A|G) \]
\[ P(B|AG) \]

\[ \rho_G \geq 0, \quad Tr \rho_G = 1 \]

\[ \rho_B = Tr_A [\rho_B|A \rho_A] \]

[Leifer&Spekkens]
Ansatz II: modification of classical causal models

general: two Hilbert spaces

Hilbert space

intervention

directed acyclic graph

conditionals

belief propagation

quantum (causal) conditional

density operator

quantum belief propagation

general: two Hilbert spaces

quantum (causal) conditional
General causal maps and the relation to process matrices

Causal maps

Local interventions described by

\[ \rho_{o|i}^{sr} \in \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_o) : \tilde{\rho}_{o}^{sr} = \text{Tr}_i [ \rho_{o|i}^{sr} \rho_i ] \]

\[ \rho_{o|i}^{T(i)} \geq 0, \quad \text{Tr}_o \left[ \sum_r \rho_{o|i}^{sr} \right] = I_i \]
General causal maps and the relation to process matrices

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Probabilities of outcomes given settings

\[ P(r, q|s, t) = \text{Tr} \left[ \rho_{Ao|Ai}^{sr} \otimes \tilde{\rho}_{Bo|Bi}^{iq} \tau_{Ai, Bi|Ao, Bo} \right] \]
General causal maps and the relation to process matrices

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\[ \rho_{o|i}^{sr} \in \mathcal{L} (\mathcal{H}_i \otimes \mathcal{H}_o) : \tilde{\rho}_o^{sr} = Tr_i [\rho_{o|i}^s \rho_i] \]

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Probabilities of outcomes given settings

\[ P(r, q|s, t) = Tr [\rho_{Ao|Ai}^{sr} \otimes \tilde{\rho}_{Bo|Bi}^{iq} \tau_{Ai, Bi|Ao, Bo}] \]

Mathematical properties

\[ \tau_{Ai, Bi|Ao, Bo}^{T(Ai, Bi)} \geq 0 \]

\[ Tr \left[ \Sigma_{r, q} \rho_{Ao|Ai}^{sr} \otimes \tilde{\rho}_{Bo|Bi}^{iq} \tau_{Ai, Bi|Ao, Bo} \right] = 1 \]

\[ \forall \rho_{Ao|Ai}, \tilde{\rho}_{Bo|Bi} \]
General causal maps and the relation to process matrices

Causal maps

Local interventions described by
\[ \rho_{o|i}^{sr} \in \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_o): \tilde{\rho}_o^{sr} = \text{Tr}_i[\rho_{o|i}^{sr} \rho_i] \]
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\[ P(r, q|s, t) = \text{Tr}[\rho_{Ao|Ai}^{sr} \otimes \tilde{\rho}_{Bo|Bi}^{iq} \tau_{Ai, Bi|Ao, Bo}] \]

Mathematical properties
\[ \tau_{Ai, Bi|Ao, Bo}^{T(Ai, Bi)} \geq 0 \]
\[ \text{Tr} \left[ \Sigma_{r, q} \rho_{Ao|Ai}^{sr} \otimes \tilde{\rho}_{Bo|Bi}^{iq} \tau_{Ai, Bi|Ao, Bo} \right] = 1 \]
\[ \forall \rho_{Ao|Ai}, \tilde{\rho}_{Bo|Bi} \]

Process matrices

Local interventions described by
\[ M_{sr}^{\text{tr}} \in \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_o): \text{Tr}_i[M_{io}^{sr} \rho_i^{T}] = \tilde{\rho}_o^{sr} \]
\[ M_{sr}^{tr} \geq 0, \quad \text{Tr}_o[\Sigma_r M_{io}^{sr}] = I_i \]

Probabilities of outcomes given settings
\[ P(r, q|s, t) = \text{Tr}[M_{Ai, Ao}^{sr, tr} \otimes \tilde{M}_{Bi, Bo}^{iq, tr} W] \]

Mathematical properties
\[ W \geq 0 \]
\[ \text{Tr} \left[ \Sigma_{r, q} M_{Ai, Ao}^{sr} \otimes \tilde{M}_{Bi, Bo}^{iq} W \right] = 1 \quad \forall M, \tilde{M} \]
Inference for quantum systems

Case 1: retrodiction of cause given effect

Classical Bayesian inversion:

\[
P(Z, Q) = P(Z|Q)P(Q)
\]

\[
\rightarrow P(Q|Z) = \frac{P(Z, Q)}{P(Z)}
\]
Inference for quantum systems

Case 1: retrodiction of cause given effect

Classical Bayesian inversion:

\[
P(Z, Q) = P(Z|Q)P(Q) \\
\rightarrow P(Q|Z) = \frac{P(Z, Q)}{P(Z)}
\]

Quantum version:

\[
\rho_{Z,Q} = \frac{1}{\sqrt{2}} \rho_{Q|Z} \rho_{Z|Q} \frac{1}{\sqrt{2}} = \rho_{Z|Q} \ast \rho_Q
\]

\[
\rho_{Q|Z} = \frac{1}{\sqrt{2}} \bigg( \rho_{Q} \otimes \rho_{Z}^{-\frac{1}{2}} \bigg) \rho_{Z|Q} \bigg( \rho_{Q} \otimes \rho_{Z}^{-\frac{1}{2}} \bigg)
\]

Note: same mathematical properties as causal conditionals

\[
\rho_{Q|Z}^{T(Z)} \geq 0
\]

[Leifer&Spekkens, Horsman et al]
Inference for quantum systems

Case 1: retrodiction of cause given effect

\[ P(Z, Q) = P(Z|Q) P(Q) \]
\[ \rightarrow P(Q|Z) = \frac{P(Z, Q)}{P(Z)} \]

Quantum version:

\[ \rho_{Z,Q} = \rho_{Q}^T \rho_{Z|Q} \rho_{Z}^{-\frac{1}{2}} = \rho_{Z|Q} \rho_{Q} \]
\[ \rho_{Q|Z} = \left( \frac{1}{2} \rho_{Q} \rho_{Z}^{-\frac{1}{2}} \right) \rho_{Z|Q} \left( \frac{1}{2} \rho_{Q} \rho_{Z}^{-\frac{1}{2}} \right) \]

Note: same mathematical properties as causal conditionals

\[ \rho_{Q|Z}^{T(Z)} \geq 0 \]

[Leifer&Spekkens, Horsman et al]

Case 2: inference via a common cause

\[ P(C|Z) = \sum_q P(C, Q=q|Z) \]
\[ = \sum_q P(C|Q=q) P(Q=q|Z) \]
Inference for quantum systems

Case 1: retrodiction of cause given effect

\[ P(Z,Q) = P(Z|Q)P(Q) \]
\[ \Rightarrow P(Q|Z) = \frac{P(Z,Q)}{P(Z)} \]

Classical Bayesian inversion:

Quantum version:

\[ \rho_{Z,Q} = \rho_Q^* \rho_{Z|Q} \rho_Q \]
\[ \rho_{Q|Z} = \left( \rho_Q^{\frac{1}{2}} \otimes \rho_Z \right) \rho_{Z|Q} \left( \rho_Q^{\frac{1}{2}} \otimes \rho_Z^{-\frac{1}{2}} \right) \]

Note: same mathematical properties as causal conditionals

\[ \rho_{Q|Z}^{T(Z)} \geq 0 \]

[Leifer&Spekkens, Horsman et al, Ried et al]

Case 2: inference via a common cause

Classically,

\[ P(C|Z) = \sum_q P(C,Q=q|Z) = \sum_q P(C|Q=q)P(Q=q|Z) \]

Quantum version:

\[ \rho_{C|Z} = Tr_Q[\rho_{C|Q} \rho_{Q|Z}] \]

Note:

• different mathematical properties

\[ \rho_{C|Q}^{T(Q)} \geq 0, \ \rho_{Q|Z}^{T(Z)} \geq 0 \Rightarrow \rho_{C|Z} \geq 0 \]

→ conditionals reflect causal structure

• no simple form for joint state:

\[ \rho_{C|Q} \ast (\rho_{Z|Q} \ast (\rho_Q \otimes \rho_Z^{-1})) \neq (\rho_{C|Q} \ast \rho_{Z|Q}) \ast (\rho_Q \otimes \rho_Z^{-1}) \neq ... \]
Given an operator relating several quantum systems,
- Can it be decomposed into separate causal relations?
- What kinds of causal relations can there be?
- How to classify the possible causal relations?
Causal Structure in a Quantum World

Causal Loops
Causal loops revisited

Case 1

\[ W = I + Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
Causal loops revisited

Case 1

\[ W = I + Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Alice measures in the Z basis, then flips the bit:

\[
\begin{align*}
M^{00} &= |0\rangle_i \otimes |1\rangle_o \\
M^{01} &= |1\rangle_i \otimes |0\rangle_o \\
M^{10} &= |1\rangle_i \otimes |0\rangle_o \\
M^{11} &= |0\rangle_i \otimes |1\rangle_o
\end{align*}
\]

(single \( s \))

\[
\Rightarrow \sum_r P(r|s=0) = Tr \left[ \sum_r M_{io}^{0r} W_{io} \right] = 0
\]

\(\Rightarrow\) Does not satisfy physical constraints.
Causal loops revisited

Case 2

\[ W = (I + Z \otimes Z)_{AoBi} \otimes (I + Z \otimes Z)_{BoAi} \]
Causal loops revisited

Case 2

\[ W = (I + Z \otimes Z)_{A_0 B_i} \otimes (I + Z \otimes Z)_{B_0 A_i} \]
Causal loops revisited

Case 2

\[ W = (I + Z \otimes Z)_{AoBi} \otimes (I + Z \otimes Z)_{BoAi} \]

Alice measures and flips, Bob just measures:

\[
\begin{align*}
M^{00} &= |0\rangle \langle 0|_{Ai} \otimes |1\rangle \langle 1|_{Ao} \\
M^{01} &= |1\rangle \langle 1|_{Ai} \otimes |0\rangle \langle 0|_{Ao} \\
\tilde{M}^{00} &= |0\rangle \langle 0|_{Bi} \otimes |0\rangle \langle 0|_{Bo} \\
\tilde{M}^{01} &= |1\rangle \langle 1|_{Bi} \otimes |1\rangle \langle 1|_{Bo}
\end{align*}
\]

\[ \sum_{r,q} P(rq|st) = Tr \left[ \sum_{r,q} M^{sr}_{Ai,Ao} \otimes \tilde{M}^{tq}_{Bi,Bo} W \right] = 0 \]

\[ \Rightarrow \quad \text{Does not satisfy physical constraints.} \]

Note: Constraints on probabilities rule out (at least some types of) causal loops.
Case 3

\[
W = \frac{1}{2} W_{B<A} + \frac{1}{2} W_{A<B} \\
= \frac{1}{8} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} \right) + \frac{1}{8} \left( I + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{A_i} \otimes Z_{Ao} \right)
\]
Case 3

\[ W = \frac{1}{2} W_{B<A} + \frac{1}{2} W_{A<B} \]
\[ = \frac{1}{8} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} \right) + \frac{1}{8} \left( I + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{A_i} \otimes Z_{Ao} \right) \]

check positivity:

\[ \lambda_{B<A} = \frac{1}{4} \left( 1 \pm \frac{1}{\sqrt{2}} \right) ; \quad \lambda_{A<B} = \frac{1}{4} \left( 1 \pm \frac{1}{\sqrt{2}} \right) \]
Causal loops revisited

Case 3

\[ W = \frac{1}{2} W_{B<A} + \frac{1}{2} W_{A<B} \]
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check positivity:
\[ \lambda_{B<A} = \frac{1}{4} \left( 1 \pm \frac{1}{\sqrt{2}} \right) \]
\[ \lambda_{A<B} = \frac{1}{4} \left( 1 \pm \frac{1}{\sqrt{2}} \right) \]

check normalization:
\[ Tr \left[ \sum_{r,q} M_{Ai, Ao}^{sr} \otimes \tilde{M}_{Bi, Bo}^{tq} W \right] = 1 \quad \forall M, \tilde{M} \]
\[ \left( M^{sr} \geq 0, \quad Tr_{o} \left[ \sum_{r} M_{io}^{sr} \right] = I_{i} \right) \]
Causal loops revisited

**Case 4**

\[
W = \frac{1}{4} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right) \\
= \frac{1}{8} \left( I + \frac{2}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} \right) + \frac{1}{8} \left( I + \frac{2}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right)
\]

[Oreshkov et al]
Causal loops revisited

Case 4

\[ W = \frac{1}{4} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right) \]
\[ = \frac{1}{8} \left( I + \frac{2}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} \right) + \frac{1}{8} \left( I + \frac{2}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right) \]

- naive attempt at convex decomposition fails:
  \[ \lambda_{B<A} = \frac{1}{4} (1 \pm \sqrt{2}) ; \quad \lambda_{A<B} = \frac{1}{4} (1 \pm \sqrt{2}) \]
Causal loops revisited

Case 4

\[ W = \frac{1}{4} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right) \]
\[ = \frac{1}{8} \left( I + \frac{2}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} \right) + \frac{1}{8} \left( I + \frac{2}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right) \]

- naive attempt at convex decomposition fails:
  \[ \lambda_{B < A} = \frac{1}{4} (1 \pm \sqrt{2}) \quad ; \quad \lambda_{A < B} = \frac{1}{4} (1 \pm \sqrt{2}) \]

- yet \( W_q \) is a valid process:
  \[ \lambda_W \in \left( 0, \frac{1}{4} \right] \geq 0 \]
  \[ Tr \left[ \sum_{r,q} M^{sr} \otimes \tilde{M}^{tq} W \right] = 1 \quad \forall \ M, \tilde{M} \]
The causal separability game

The task:
- if \( j=0 \), Alice must signal Bob: return \( q=s \)
- if \( j=1 \), Bob must signal Alice: return \( r=t \)
The causal separability game

The task:
- if $j=0$, Alice must signal Bob: return $q=s$
- if $j=1$, Bob must signal Alice: return $r=t$

Success probability given a fixed causal order:

$p_{\text{suc}} \leq \frac{3}{4}$

Success probability using

$$W = \frac{1}{4} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right)$$

[Oreshkov et al]
The causal separability game

The task:
- if \(j=0\), Alice must signal Bob: return \(q=s\)
- if \(j=1\), Bob must signal Alice: return \(r=t\)

Success probability given a fixed causal order:

\[
p_{\text{suc}} \leq \frac{3}{4}
\]

Success probability using

\[
W = \frac{1}{4} \left( I + \frac{1}{\sqrt{2}} Z_{Bo} \otimes Z_{Ai} + \frac{1}{\sqrt{2}} Z_{Bi} \otimes X_{Ai} \otimes Z_{Ao} \right)
\]

- if \(j=0\), Alice measures in \(Z\) basis
- if \(j=1\), Alice measures in \(X\) basis

\[
\Rightarrow p_{\text{suc}} = \frac{2 + \sqrt{2}}{4}
\]

[Oreshkov et al]
Causal witnesses

More generally, one can find observables $S$ such that, for all $W^{sep}$ of the form

$$W^{sep} = q W_{B < A} + (1 - q) W_{A < B}$$

it holds that

$$Tr[S W^{sep}] \geq 0$$

Compare with witnesses of entanglement:

$$Tr[\tilde{S} \rho^{sep}] \geq 0 \quad \forall \rho^{sep} = \sum_j q_j \rho_A^j \otimes \rho_B^j$$
The quantum switch

[Chiribella et al]
The quantum switch

- prepare a superposition of the control qubit
- post-select on still having a superposition afterwards

[Chiribella et al]
Applications of indefinite causal order

Perfect discrimination of no-signalling channels via quantum superposition

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A no-signalling channel transforming quantum systems in Alice’s and Bob’s local laboratories with two different causal structures: \( A \preceq B \) Alice’s output causally precedes Bob’s input, and \( B \preceq A \) Bob’s output causally precedes Alice’s input. Here I prove that two no-signalling channels distinguishable in any ordinary quantum circuit can become perfectly distinguishable if they are superimposed. A superposition of circuits with different causal structures.

Testing permutation of unitaries: polynomial reduction in query complexity

[Colnaghi et al, Araújo et al]

Exchange evaluation game: exponential advantage in communication complexity

[Guérin et al]
Experimental superposition of orders of quantum gates

[Diagram of experimental setup with various optical components and labels such as Laser, APD, BP, QWP, HWP, DM, LP, FC, and abbreviations for optical devices like Dichroic Mirror, Beamsplitter, etc.]
Experimental superposition of orders of quantum gates

How to count the number of uses of the black boxes?

- points in spacetime
- internal degree of freedom
Superposition of causal structures using general relativity

\[ \tau_A = 2 \]

Alice

\[ \tau_B = 2 \]

Bob

\[ \tau_A = 2 \]

Alice

\[ \tau_B = 2 \]

Bob

[Feix&Brukner]
Superposition of causal structures using general relativity

A precedes B

\[ \tau_A = 2 \]

\[ \tau_B = 2 \]

Alice

Bob

B precedes A

\[ \tau_A = 2 \]

\[ \tau_B = 2 \]

Alice

Bob

required parameters:
for a spatial superposition of \( \Delta x = 10^{-3} \text{ m} \), \( M = 1 \text{ g} \) need time resolution of \( 10^{-27} \) experimentally feasible:
spatial superposition of \( \Delta x = 10^{-6} \text{ m} \), \( M \approx 10^{-21} \text{ g} \), time resolution \( 10^{-18} \)

[Feix&Brukner]
Causal Structure in a Quantum World

Combinations of causal relations
Testbed for combining causal relations within a well-defined causal order:

- Common cause: $P(A, B)$
- Cause-effect: $P(B|A)$, $P(A)$
Mixing common-cause and cause-effect relations

purely common-cause (CC)

purely cause-effect (CE)

coin toss J
Mixing common-cause and cause-effect relations

\[ P(B|A,\lambda,\text{heads}) = P(B|\lambda) \]
\[ \Rightarrow \] purely common-cause

\[ P(B|A,\lambda,\text{tails}) = P(B|A) \]
\[ \Rightarrow \] purely cause-effect
Mixing common-cause and cause-effect relations

Probabilistic mixture:

\[ P(B|A,\lambda,\text{heads}) = P(B|\lambda) \]
\[ \Rightarrow \text{purely common-cause} \]

\[ P(B|A,\lambda,\text{tails}) = P(B|A) \]
\[ \Rightarrow \text{purely cause-effect} \]

Physical mixture:

\[ P(B|A,\lambda,\text{heads}) = P(B|A,\lambda) \]
\[ P(B|A,\lambda,\text{tails}) = P(B|A,\lambda) \]
\[ \Rightarrow \text{both CC and CE} \]

Note: both of these ways of combining causal relations are \textit{classical}. 
Probabilistic mixture:

Physical mixture:
How to detect a combination of two causal influences?
Berkson's paradox

- Faculty position
- Teaching ability
- Research ability

All applicants

- Research
- Teaching
Berkson's paradox

- faculty position
- teaching ability
- research ability
Berkson's paradox (extended edition)

- faculty position
- teaching ability
- citations
- research ability

Graph showing data points for all applicants, faculty, and relationships between teaching ability, research ability, and citations.
Berkson-type induced correlations:
- classical
  \[ P(\text{CD}|\text{B}) \]
- quantum
  \[ \{ E_B^b \} \Rightarrow \{ \tau_{CD}^b \} \]
Distinguishing combinations of causal structures

purely cause-effect
or purely common-cause:

\[
P(CD | B)
\]

independent

probabilistic mixture:

\[
(1-q) + q \Rightarrow
\]

weak correlations
physical mixture (not probabilistic):

$$P(CD|B)$$

strong correlations

for example

$$B = D \oplus \lambda$$

$$C = \lambda$$
physical mixture (not probabilistic):

\[ P(CD \mid B) \]

strong correlations

for example

\[
\begin{align*}
B &= D \oplus \lambda \\
C &= \lambda
\end{align*}
\]

then

\[
\begin{align*}
B &= 0 \\
\Rightarrow C &= D
\end{align*}
\]

perfect correlation
physical mixture (not probabilistic):

\[ P(CD|B) \] 

\[ B = D \oplus \lambda \]
\[ C = \lambda \]

then

\[ B = 0 \]
\[ \Rightarrow C = D \]

perfect correlation

⇒ Conversely, strong correlations rule out a probabilistic mixture.
physical mixture (not probabilistic):

intrinsically quantum combination:

\[ P(CD|B) \]

\[ \tau(CD|B) \neq \sum_i \rho_C^{(i)} \otimes \rho_D^{(i)} \]

\[ B = D \oplus \lambda \]
\[ C = \lambda \]

then

\[ B = 0 \]
\[ \Rightarrow C = D \]

perfect correlation
Two quantum variables with tunable causal relation

coupling:

local

swap

cause-effect

common cause
cause-effect

coupling: \[ U = \cos \frac{\theta}{2} 1 + i \sin \frac{\theta}{2} S \]

\[ \theta = 0 \]

\[ \theta = \pi \]

cause-effect

common cause
Berkson-type induced entanglement

post-selection \( \rho_B = |0\rangle\langle 0| \)

induced state

\[
\rho^{(0)}_{D\lambda} = U^\dagger \left( \rho_B \otimes \frac{1}{2} \mathbf{1} \right) U \\
= \frac{1}{2} U^\dagger [ |00\rangle\langle 00| + |01\rangle\langle 01| ] U \\
= \frac{1}{2} |00\rangle\langle 00| + \frac{1}{2} |\Psi^-\rangle\langle \Psi^-| 
\]
$U = \cos \frac{\theta}{2} \mathbf{1} + i \sin \frac{\theta}{2} \mathbf{S}$

[MacLean et al, Nat Comm 8]
Induced negativity
witnesses non-classical causal structure
Take-home messages and open questions
Take-home messages and open questions

Causal models framework: mathematical and conceptual toolbox

- classical: rigorous definition of causation, methods for inferring causal relations and deriving predictions from this information
- quantum: description of relations among quantum systems in terms of operators that can (at least in part) be interpreted causally, distinction between causation and inference

Causal models provide a clear language and context for analysing many counter-intuitive phenomena in quantum mechanics, such as the apparent retrocausality in delayed-choice experiments, propagation outside the lightcone in quantum field theory, and the tangle of assumptions the lead to Bell inequalities.

The conjunction of all the principles that hold in classical causal models (Reichenbach, no fine-tuning etc) is at odds with the predictions of quantum mechanics. However, it is difficult to determine which of these principles are violated. More work is needed to develop a convincing, consistent account of causality that allows one to give up any of these principles.
Take-home messages and open questions

Two proposals for how quantum mechanics might handle causal loops:
• allow generic causal loops but give up linearity, which leads to unusual information flow
• preserve linearity but allow only a restricted class of causal loops

Non-classical causal relations
• There are a few concrete examples of such scenarios, but a systematic account of all the possibilities is still outstanding.
• Some have been realized experimentally, but all experiments so far were embedded in a background spacetime with well-defined causal order. It would be interesting to overcome this limitation.
• Non-classical causal structures are known to be resources for certain tasks. What other advantages can be extracted from these phenomena and what fundamental insights does this entail?
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