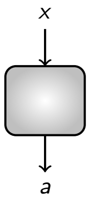
Kochen-Specker contextuality

Lecture 1

Ana Belén Sainz

Solstice of Foundations summer school – ETH Zurich
19/06/2017

Kochen-Specker contextuality



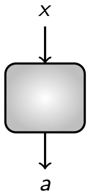
$p(a|x)$, $\langle x \rangle$

Kochen-Specker contextuality

Some quantum statistics are incompatible with

- deterministic assignments of values to all the observables
- satisfying the compatibility relations inherited from quantum mechanics

(or mixtures of these models.)



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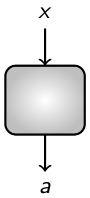

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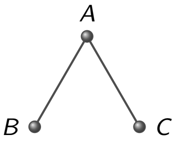

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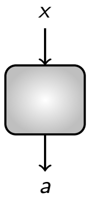

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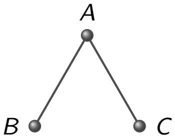
$p(ab|AB)$



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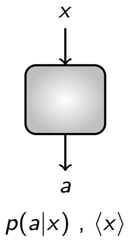

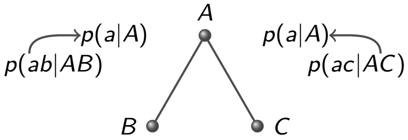

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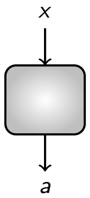

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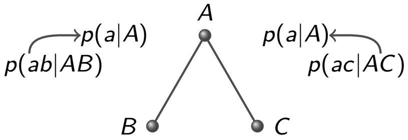
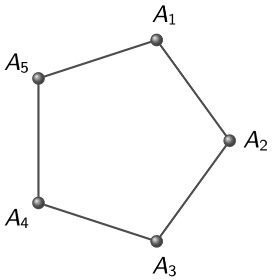


Non-contextual, deterministic, hidden-variable model:

Λ: $p(a|A\,,\Lambda) \in \{0, 1\}$.

Klyachko-Can-Binicioğlu-Shumovsky (KCBS)



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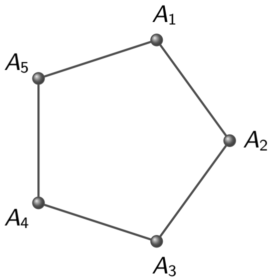

Non-contextual hidden variable model (deterministic): $\lambda : A_i \to \pm 1$

Distribution over hidden variables: $p(\lambda)$

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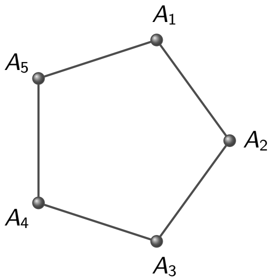

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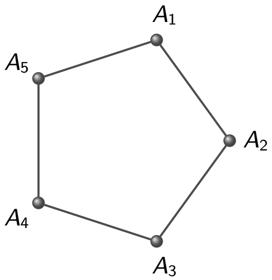

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$$\text{st} \quad a_i = \pm 1 \quad \forall\, i\,.$$

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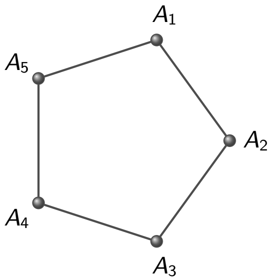

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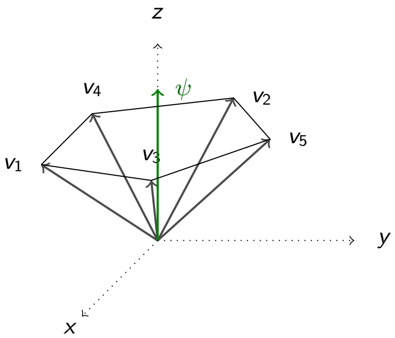
KCBS: quantum violation



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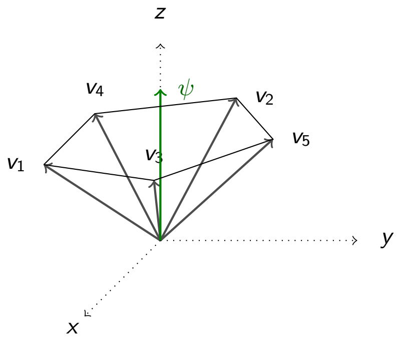

$$A_k = 2|v_k\rangle\langle v_k| - \mathbb{1}$$

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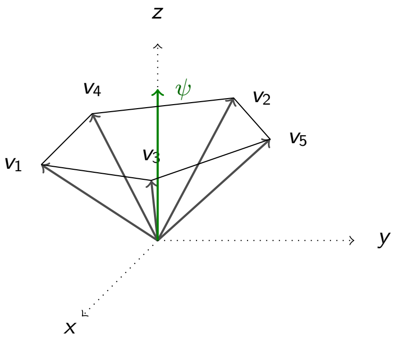

$$A_k = 2|v_k\rangle\langle v_k| - \mathbb{1}$$

$$K = 5 - 4\sqrt{5} \sim -3.94427$$

Quantum mechanics violates the KCBS inequality

State-independent contextuality

Peres-Mermin square: nine observables $\{A, B, C, a, b, c, \alpha, \beta, \gamma\}$

$A = \sigma_z \otimes \mathbb{1}$	$B = \mathbb{1} \otimes \sigma_z$	$C = \sigma_z \otimes \sigma_z$
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Quantum mechanics violates the inequality for all quantum states.

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Cabello, Severini and Winter $\rightarrow$ inequalities from the compatibility structure of events

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What is an event? $\rightarrow$ measured context and obtained outcomes

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What is an event? $\rightarrow$ measured context and obtained outcomes

Example: KCBS

$$\{(a_i, a_{i+1}|A_i, A_{i+1}) \,|\, a_i, a_{i+1} = \pm 1\,,\ 1 \leq i \leq 5\}$$

KCBS: second formulation

Five yes/no questions: $\{P_i\,,\ 1 \leq i \leq 5\}$,

- P_i and P_{i+1} are compatible,
- P_i and P_{i+1} are exclusive. That is, they can't be both simultaneously answered with 'yes'.

What is the maximum number of 'yes' that we can obtain?

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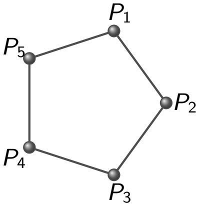

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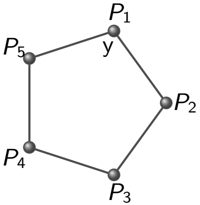

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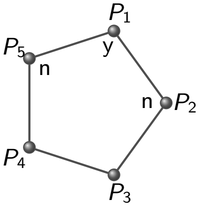

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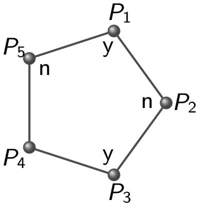

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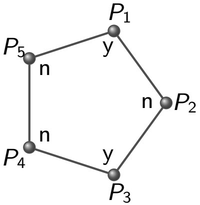

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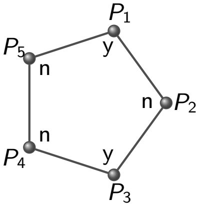

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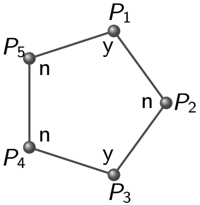

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First formulation of KCBS?

$$A_i = 2P_i - 1\,, \quad \Rightarrow \quad \langle A_i A_{i+i} \rangle = -2\langle P_i \rangle - 2\langle P_{i+1} \rangle + 1\,,$$

$$\sum_i \langle A_i\, A_{i+1} \rangle \underset{\text{NCHV}}{\geq} -3\,.$$

KCBS $\rightarrow$ Inequalities from graphs

Graph:

- Vertices: Events of the scenario.
  $\{(0|P_i), (1|P_i)\}_i$
- Edges: join exclusive events

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Equip the graph's vertices with weights (G, w): $w_{(1|P_i)} = \alpha_i, \; w_{(0|P_i)} = \beta_i$

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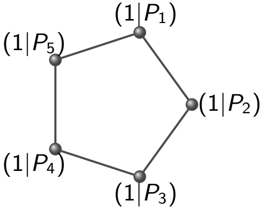

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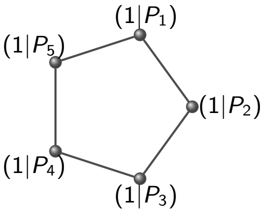

Independence number of the pentagon:
$\alpha = 2$

$$\sum_{i=1}^{5} \langle P_i \rangle \underset{\text{NCHV}}{\leq} 2 \,.$$

KCBS $\rightarrow$ Inequalities from graphs

Quantum violation?

Weighted Lovász number of (G, w): $\vartheta(G, w)$

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"Orthogonal representation": $|\Psi\rangle$, $\{|\phi_v\rangle\}_v$

- unit vectors
- $\langle \phi_v | \phi_u \rangle = 0$ if $u \underset{G}{\sim} v$.

$$\vartheta(G, w) = \sum_{v \in V} w(v) \, |\langle \phi_v | \Psi \rangle|^2 \, .$$

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If $|\Psi\rangle \rightarrow$ quantum state, $|\phi_v\rangle\langle\phi_v| \rightarrow$ projector associated to answer v: quantum correlations!

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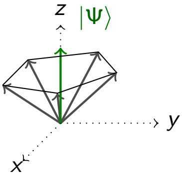
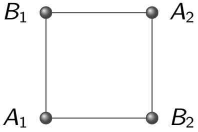


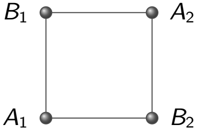
$$\vartheta(G, w) = \sqrt{5} > 2$$

Bell scenarios: CHSH



Compatible measurements: $\{A_i, B_j\}$

Bell scenarios: CHSH



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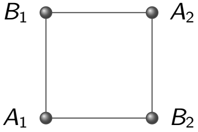
Bell scenarios: CHSH



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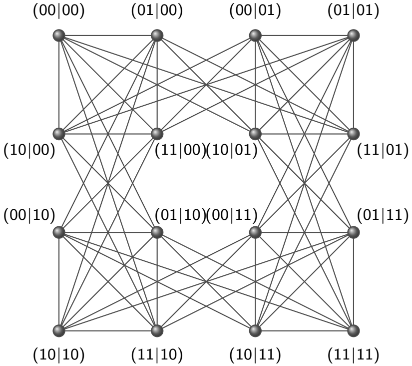
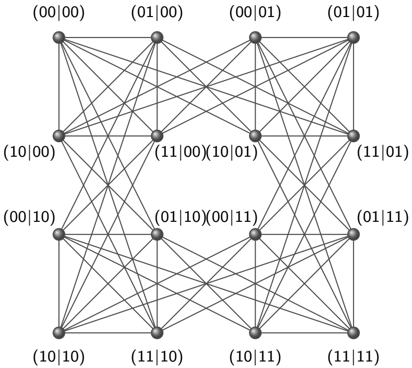
Local Orthogonality: two events are orthogonal if there is a party that has chosen the same measurement in both, but obtained different outcomes.

Example: $(00|00) \perp (10|01)$ but $(00|00) \not\perp (01|01)$.

Bell scenarios: CHSH



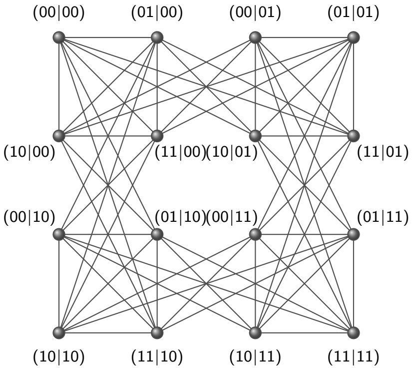
Bell scenarios: CHSH



CHSH inequality:

$$\sum_{\substack{ab \\ a=b}} p(ab|00) + \sum_{\substack{ab \\ a=b}} p(ab|10) + \sum_{\substack{ab \\ a=b}} p(ab|01) + \sum_{\substack{ab \\ a\neq b}} p(ab|11) \underset{\text{NCHV}}{\leq} 3$$

Bell scenarios: CHSH



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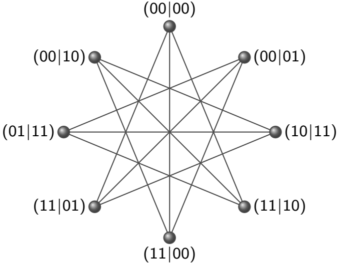
Equip the graph with weights: $w(ab|xy) = \delta_{a\oplus b = xy}$

Bell scenarios: CHSH



Eight-vertex circulant $(1, 4)$ graph: $Ci_8(1, 4)$

$\alpha(G, w) = 3$, $\vartheta(G, w) = 2 + \sqrt{2}$

CSW: limitations

For Bell scenarios, $\vartheta(G, w)$ is only an upper bound to Tsirelson's bound.

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A true quantum model in a Bell scenario must satisfy the following constraints:

(i) Normalisation of probabilities: $\sum_{v \in e} |\langle \phi_v | \Psi \rangle|^2 = 1$, for every complete measurement e.
Example: $e = \{(ab|xy) : a, b = 0, 1\}$

(ii) Normalisation of the von Neumann measurements: $\sum_{v \in e} |\phi_v\rangle\langle\phi_v| = \mathbb{1}$, for every complete measurement e.

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Example: I_{3322} Bell inequality.

- $\vartheta(G, w) \sim 0.4114$
- $\vartheta(G, w)$ constrained via (i): bound$= 0.25147$
- quantum bound< 0.2508755

Summary of today

- Kochen-Specker contextuality

  S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).

- KCBS example

  A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. 101, 020403 (2008).

- State-independent contextuality

  N.D.Mermin, Phys.Rev.Lett. 65, 3373-6 (1990).

  A.Peres, Phys. Lett. A 151, 107-8 (1990).

- Inequalities from hypergarphs: CSW approach
  - KCBS
  - CHSH Bell scenario
  - Limitations: I_{3322}

  A. Cabello, S. Severini, A. Winter, arXiv:1010.2163