Kochen-Specker contextuality

Lecture 2

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Solstice of Foundations summer school – ETH Zurich
20/06/2017

Yesterday...

- Kochen-Specker contextuality
- KCBS example
- State-independent contextuality
- Inequalities from hypergarphs: CSW approach
  - KCBS
  - CHSH Bell scenario
  - Limitations: I_{3322}

Scenarios with operational equivalences

Acín-Fritz-Leverrier-Sainz → Dual approach to CSW

CSW: inequalities, AFLS: probabilisitc models

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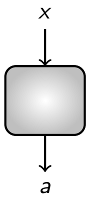
- Set of measurements
- Set of outcomes



Events: $(a|x)$

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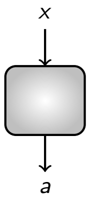

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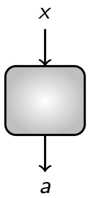

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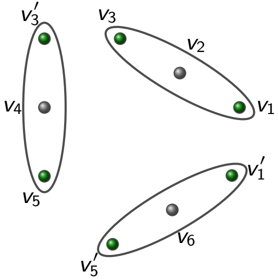

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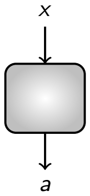

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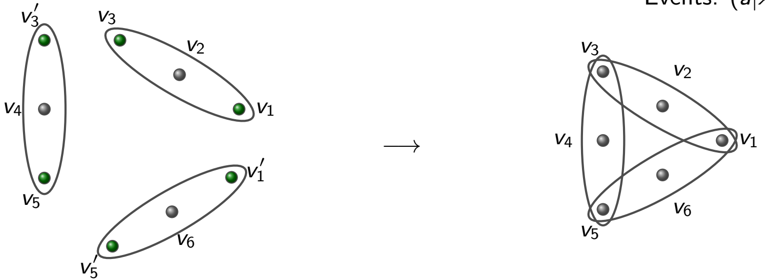
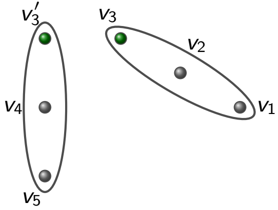


Scenarios with operational equivalences

Example: two projective measurements.



(1) $\{\Pi_1, \Pi_2, \Pi_3\}$ associated to outcomes $\{v_1, v_2, v_3\}$.

(2) $\{\Pi_3, \Pi_4, \Pi_5\}$ associated to outcomes $\{v_3', v_4, v_5\}$.

$\sum_{i=1}^{3} \Pi_i = \mathbb{1} = \sum_{i=3}^{5} \Pi_i$.

Born's rule: $p(v_3) = \mathrm{tr}\,\{\Pi_3\,\rho\} = p(v_3') \quad \forall\,\rho$

Scenarios with operational equivalences

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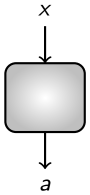
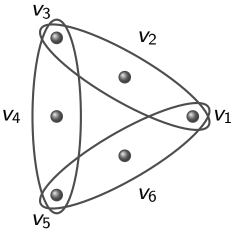


Hypergraph:

- Vertices → events – measurement outcome
- Hyperedges → complete measurements – set of outcomes



Probabilistic models

Probabilistic model $\rightarrow$ outcome statistics respecting operational equivalences

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Probabilistic model

Given $H = (V, E)$, $\quad p : V \rightarrow [0, 1]$

such that $\qquad \mathcal{G}(H)$

$\sum_{v \in e} p(v) = 1 \quad \forall\, e \in E$

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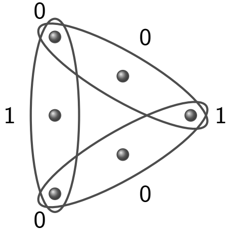

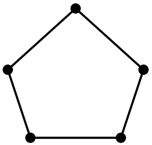

Classical probabilistic models

Classical $\rightarrow$ deterministic noncontextual hidden variables

There are ‘hidden variables’ that determine (with certainty) which measurement outcome happens, and we only observe an average over them, according to the preparation of our physical system.

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Classical model

A probabilistic model $p : V \rightarrow [0,1]$ is classical iff

$p(v) = \sum_\lambda q_\lambda p_\lambda(v)$,

where $\sum_\lambda q_\lambda = 1$, and p_λ is a deterministic model for each λ.

$\mathcal{C}(H)$

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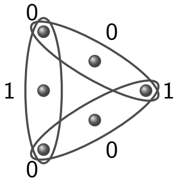

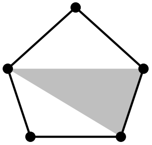

Quantum probabilistic models

Quantum models

A probabilistic model $p : V \to [0,1]$ is quantum if
$\exists \quad \mathcal{H}, \quad \rho, \quad \{P_v : v \in V\}$

$\sum_{v \in e} P_v = \mathbb{1}_{\mathcal{H}} \qquad \forall\, e \in E$

$\mathcal{Q}(H)$

$p(v) = \mathrm{tr}(\rho P_v)$

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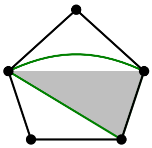
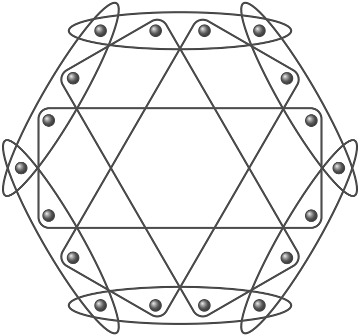


State-independent contextuality



Nine measurements of four possible outcomes each.

$$\mathcal{C}(H) = \emptyset \text{ while } \mathcal{Q}(H) \neq \emptyset$$

The non-orthogonality graph

"Two events are orthogonal if there exists a hyperedge that contains them both"

$$H(V, E) \quad \to \quad \mathrm{NO}(H)$$

The non-orthogonality graph

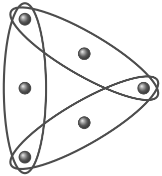
"Two events are orthogonal if there exists a hyperedge that contains them both"

$$H(V, E) \quad \rightarrow \quad \mathrm{NO}(H)$$

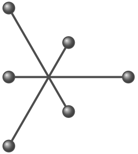
Example:

Contextuality scenario: H

Non-orthogonality graph: NO(H)



$\longrightarrow$



NO graph and probabilistic models

$$p \in \mathcal{C}(H) \quad \text{iff} \quad \alpha^* \left(\mathrm{NO}(H), p\right) = 1$$

NO graph and probabilistic models

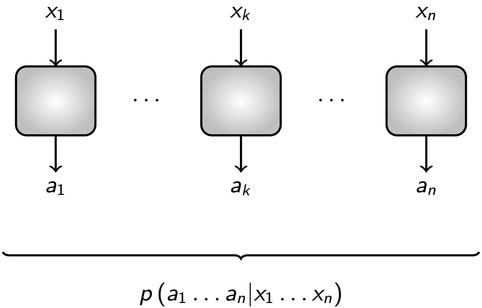
$$p \in \mathcal{C}(H) \quad \text{iff} \quad \alpha^* \left(\mathrm{NO}(H), p\right) = 1$$

Quantum models cannot be characterised by the properties of $(\mathrm{NO}(H), p)$

Example: $\exists \quad H\,, \quad H'\,, \quad p \quad$ st:

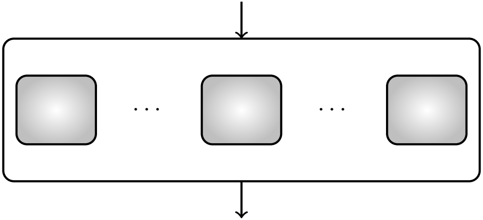
- $p \in \mathcal{Q}(H)$
- $p \in \mathcal{Q}_1(H') \setminus \mathcal{Q}(H')$
- $\mathrm{NO}(H) = \mathrm{NO}(H')$

Bell scenarios



Bell scenarios

$$x = (x_1, \ldots, x_n)$$



$$a = (a_1, \ldots, a_n)$$

$$\underbrace{\qquad\qquad\qquad\qquad}$$

$$p\left(a_1 \ldots a_n | x_1 \ldots x_n\right) \quad \rightarrow \quad p\left(a|x\right)$$

Bell scenarios

Bell scenario $\rightarrow$ events-based hypergraph?

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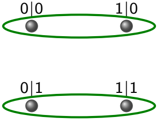
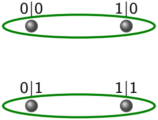
Alice



H_A

Bob



H_B

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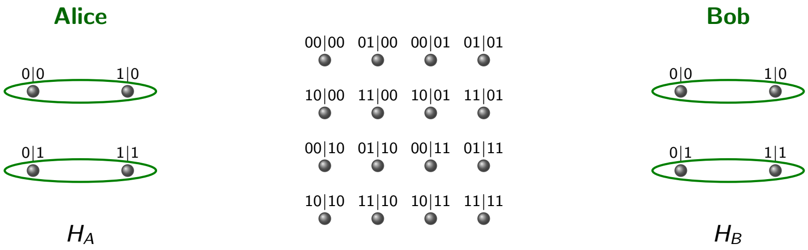
Bell scenario $\rightarrow$ events-based hypergraph?



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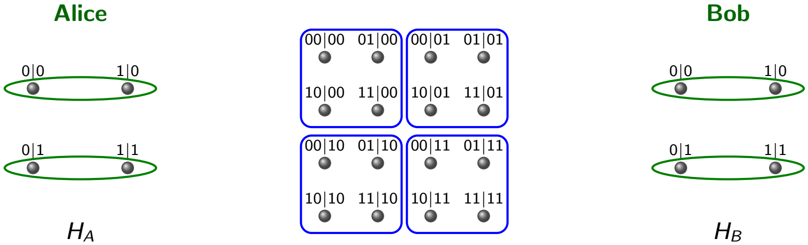

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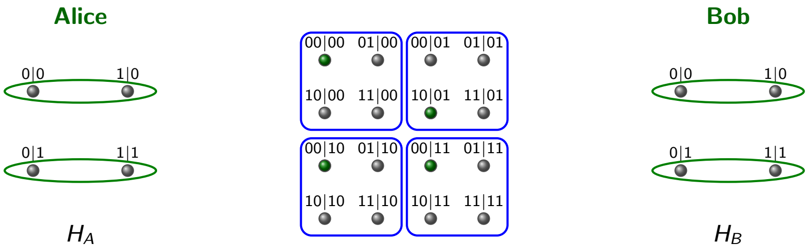
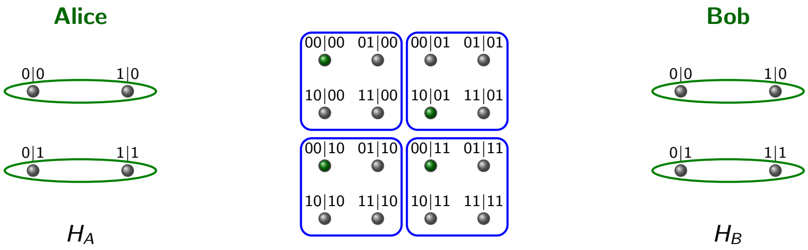


$$p : \quad (00|00) \rightarrow 1\,, \quad (10|01) \rightarrow 1\,, \quad (00|10) \rightarrow 1\,, \quad (00|11) \rightarrow 1$$

Bell scenarios

Bell scenario $\rightarrow$ events-based hypergraph?



$$p : \quad (00|00) \rightarrow 1\,, \quad (10|01) \rightarrow 1\,, \quad (00|10) \rightarrow 1\,, \quad (00|11) \rightarrow 1$$

$$p_A(0|00) = 1\,, \quad p_A(0|01) = 0.$$

This choice of hypegraph admits signalling models

Bell scenarios

Correlated measurements

- temporal order of the parties. e.g. $A \to B$
- a choice of measurement for the first party, e.g. x.
- a function $y = f(a)$ for the second party, that determines its measurement input as a function of the previous party's outcome.

Example:

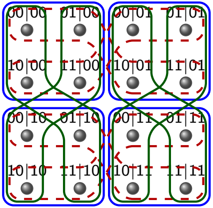
$(A \to B,\ x = 0,\ y = a)$

Correlated measurement with outcomes $\{(00|00), (01|00), (10|01), (11|01)\}$.

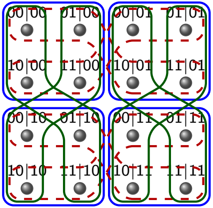
Bell scenarios

Events-based hypergraph: $\mathcal{B}_{n,m,d}$

- Vertices: events
- Egdes: correlated measurements



Bell scenarios



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- Vertices: events
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- $\mathcal{G}(\mathcal{B}_{n,m,d}) = \mathcal{NS}(n, m, d)$
- $\mathcal{C}(\mathcal{B}_{n,m,d}) = \mathcal{C}(n, m, d)$
- $\mathcal{Q}(\mathcal{B}_{n,m,d}) = \mathcal{Q}(n, m, d) \quad \longrightarrow \quad \Pi_{ab|xy} = \Pi_{a|x} \Pi_{b|y}$ with $[\Pi_{a|x}, \Pi_{b|y}] = 0$

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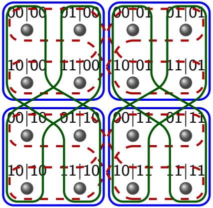

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- $\mathcal{B}_{n,m,d}$ may be computed via the "Foulis-Randall" product of the local hypergraphs.

Relation to Compatible-observables scenarios

Scenario : $(X, O, \mathcal{M})$

- X: observables
- O: outcomes
- $\mathcal{M}$: measurement cover $\rightarrow$ sets of compatible measurements
  $C \in \mathcal{M}$ measurement context

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KCBS:

$$X = \{A_i \,:\, 1 \leq i \leq 5\}$$

$$O = \{-1, 1\}$$

$$\mathcal{M} = \{\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}, \{A_4, A_5\}, \{A_5, A_1\}\}$$

Relation to Compatible-observables scenarios

$$(X, O, \mathcal{M}) \quad \longrightarrow \quad H[X]$$

- Vertices:

- Hyperedges:

Relation to Compatible-observables scenarios

$$(X, O, \mathcal{M}) \quad \longrightarrow \quad H[X]$$

- Vertices: $(s, C) : C \in \mathcal{M}, s \in O^C$

  KCBS: $(a_i\, a_{i+1} | A_i\, A_{i+1})$

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- Hyperedges: Measurement protocols

- Choose and measure an observable (A).

- Depending on the outcome,
choose a compatible observable (A').

- Measure (A'), ...

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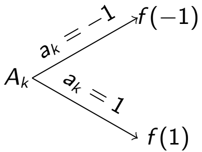

$f : \{1, -1\} \to \{A_{k-1}, A_{k+1}\}$

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- Measure (A'), ...

Contextuality bundles

Tool to demonstrate the obstruction to a NCHV model

Contextuality bundles

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Probability table:

	A_1 B_1	A_2 B_1	A_1 B_2	A_2 B_2
00	1	0	1	0
10	0	1	0	1
01	0	0	0	0
11	0	0	0	0

Contextuality bundles

Tool to demonstrate the obstruction to a NCHV model

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$\downarrow$

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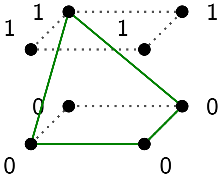

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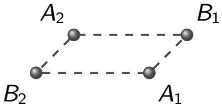

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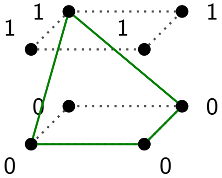

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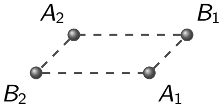

- **Local section**: assignment of 'possible' values for a given context.
- **Global section**: collection of 'compatible' local sections

Contextuality bundles

PR-box

Probability table:

	$A_1\ B_1$	$A_2\ B_1$	$A_1\ B_2$	$A_2\ B_2$
00	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
10	0	0	0	$\frac{1}{2}$
01	0	0	0	$\frac{1}{2}$
11	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

$\downarrow$

Possibility table:

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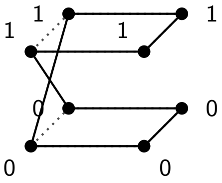

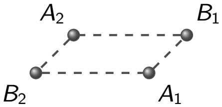

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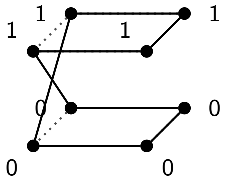
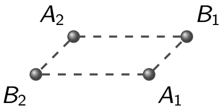
$\downarrow$

Possibility table:

	A_1 B_1	A_2 B_1	A_1 B_2	A_2 B_2
00	1	1	1	0
10	0	0	0	1
01	0	0	0	1
11	1	1	1	0





No local section can be extended to a global one

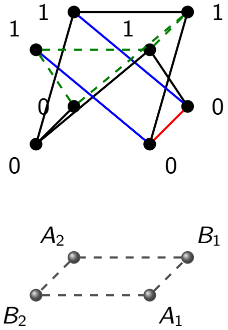
$\rightarrow$ Strong Contextuality

Contextuality bundles

A quantum example

Possibility table:

	A_1 B_1	A_2 B_1	A_1 B_2	A_2 B_2
00	1	0	0	1
10	1	1	1	1
01	1	1	1	1
11	1	1	1	0

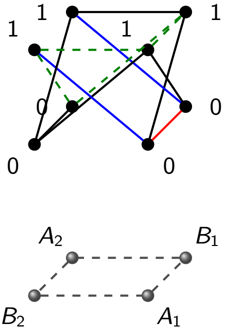


Contextuality bundles

A quantum example

Possibility table:

	$A_1\ B_1$	$A_2\ B_1$	$A_1\ B_2$	$A_2\ B_2$
00	1	0	0	1
10	1	1	1	1
01	1	1	1	1
11	1	1	1	0



Some local sections cannot be extended to a global one

→ Logical Contextuality

Summary of today

- Scenarios from operational equivalences
  - Define sets of probabilistic models
  - Graph theoretical advantages
  - Bell scenarios
  - Compatible-observables scenarios (*S. Abramsky and A. Brandenburger, New J. Phys. 13(11), 113036 (2011).*)

    A. Acín, T. Fritz, A. Leverrier, A.B. Sainz, Comm. Math. Phys. 334(2), 533-628 (2015).

- Contextuality bundles

  S. Abramsky, R. Soares Barbosa, K. Kishida, R. Lal and S. Mansfield, 24th EACSL, CSL 2015, pages 211–228, 2015.

Closing remarks

- Kochen-Specker contextuality, with focus on graph theory
- Hidden variable models, and quantum violations
- Contextuality scenarios: ‘observables with compatibility relations’, or ‘events with operational equivalences’.
- Types of probabilistic models $\rightarrow$ graph theory
  CSW: contextuality
  ALFS: set membership
- Bell scenarios as contextuality ones

Summary

- Kochen-Specker contextuality
  S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).

- KCBS example
  A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, Phys. Rev. Lett. 101, 020403 (2008).

- State-independent contextuality
  N.D.Mermin, Phys.Rev.Lett. 65, 3373-6 (1990).
  A.Peres, Phys. Lett. A 151, 107-8 (1990).

- Inequalities from hypergarphs: CSW approach
- KCBS, CHSH Bell scenario, Limitations: I_{3322}
  A. Cabello, S. Severini, A. Winter, arXiv:1010.2163

- Scenarios from operational equivalences
- Graph theoretical advantages, Bell and Compatible-observables scenarios
  A. Acín, T. Fritz, A. Leverrier, A.B. Sainz, Comm. Math. Phys. 334(2), 533-628 (2015).

- Contextuality bundles
  S. Abramsky, R. Soares Barbosa, K. Kishida, R. Lal and S. Mansfield, 24th EACSL, CSL 2015, pages 211–228, 2015.

Comments on dilation

"Dilation", a.k.a. "The Church of the larger Hilbert space":

Let $p(a|x)$ be a probabilistic model in a contextuality scenario. Let us assume that we have a realisation of it in terms of POVMs (i.e. generalised measurements); that is, a Hilbert space $\mathcal{H}$, positive semidefinite matrices $M_{a|x}$ s.t. $\sum_a M_{a|x} = \mathbb{1}_{\mathcal{H}}$ and a quantum state ρ, s.t. $p(a|x) = \mathrm{tr}\left\{M_{a|x}\,\rho\right\}$.

A *dilation* of this model is a realisation of the correlations in terms of projective measurements, i.e. a Hilbert space $\mathcal{H}'$ (possibly of larger dimension than $\mathcal{H}$), projectors $\Pi_{a|x}$ s.t. $\sum_a \Pi_{a|x} = \mathbb{1}_{\mathcal{H}'}$ and a quantum state ρ', s.t. $p(a|x) = \mathrm{tr}\left\{\Pi_{a|x}\,\rho'\right\}$.

Colloquially, the POVM operators $M_{a|x}$ are dilated into projection operators $\Pi_{a|x}$.

When can we find a dilation of a POVM model?

The mathematical counterpart of this question was addressed by Naimark and Stinespring in the context of C*-algebras. See e.g.,

M. A. Naimark. On a representation of additive operator valued set functions (Russian). Doklady Acad. Nauk SSSR, 41(5):373375, 1943

Vern Paulsen. Completely Bounded Maps and Operator Algebras. Cambridge, University Press, 2003

Comments on Dilation

Contextuality scenarios:

It is not always possible to dilate a POVM realisation of a probabilistic model, such that the dilated projectors satisfy the same compatibility relations as the original POVM elements.

C. Heunen, T. Fritz and M. L. Reyes, Phys. Rev. A 89, 032121 (2014).

Bell scenarios:

Any POVM realisation of Bell correlations has an equivalent realisation in terms or projective measurements.

Double Stinespring theorem: *T. Fritz, Rev. Math. Phys. 24(5), 1250012 (2012).* (uses C*-algebraic formulation of quantum theory)

V. Paulsen, Lecture notes on "Entanglement and Non-Locality" (Sec. 9), available at http://www.math.uwaterloo.ca/∼vpaulsen/
(does not use C*-algebras)