Kochen-Specker contextuality

Lecture 2

Ana Belén Sainz

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Yesterday...

- Kochen-Specker contextuality
- KCBS example
- State-independent contextuality

- Inequalities from hypergraphs: CSW approach
  - KCBS
  - CHSH Bell scenario
  - Limitations: $l_{3322}$
Scenarios with operational equivalences

Acín-Fritz-Leverrier-Sainz $\rightarrow$ Dual approach to CSW

CSW: inequalities, AFLS: probabilistic models
Scenarios with operational equivalences

Acín-Fritz-Leverrier-Sainz → Dual approach to CSW

CSW: inequalities, AFLS: probabilistic models

- Set of measurements
- Set of outcomes

Events: \((a|x)\)
Scenarios with operational equivalences

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- Operational equivalences $\rightarrow$ identify outcomes of different measurements: same probability

Events: $(a|x)$
Scenarios with operational equivalences

Acín-Fritz-Leverrier-Sainz → Dual approach to CSW

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Scenarios with operational equivalences

Acín-Fritz-Leverrier-Sainz → Dual approach to CSW

CSW: inequalities, AFLS: probabilistic models

- Set of measurements
- Set of outcomes
- Operational equivalences → identify outcomes of different measurements: same probability

Events: \((a|x)\)
Scenarios with operational equivalences

Example: two projective measurements.

\[(1) \{\Pi_1, \Pi_2, \Pi_3\} \text{ associated to outcomes } \{v_1, v_2, v_3\} \]

\[(2) \{\Pi_3, \Pi_4, \Pi_5\} \text{ associated to outcomes } \{v'_3, v_4, v_5\} \]

\[\sum_{i=1}^{3} \Pi_i = 1 = \sum_{i=3}^{5} \Pi_i.\]

Born’s rule: \[p(v_3) = \text{tr} \{\Pi_3 \rho\} = p(v'_3) \quad \forall \rho\]
Scenarios with operational equivalences

- Set of measurements
- Set of outcomes
- Operational equivalences → identify outcomes of different measurements: same probability

Hypergraph:
- Vertices → events – measurement outcome
- Hyperedges → complete measurements – set of outcomes
Probabilistic models

Probabilistic model $\rightarrow$ outcome statistics respecting operational equivalences
Probabilistic models

Probabilistic model $\rightarrow$ outcome statistics respecting operational equivalences

Probabilistic model

Given $H = (V, E)$, $p : V \rightarrow [0, 1]$

such that

$$\sum_{v \in e} p(v) = 1 \quad \forall e \in E$$

$\mathcal{G}(H)$
Probabilistic models

Probabilistic model $\rightarrow$ outcome statistics respecting operational equivalences

**Probabilistic model**

Given $H = (V, E), \quad p : V \rightarrow [0, 1]$

such that

$$\sum_{v \in e} p(v) = 1 \quad \forall e \in E$$

$\mathcal{G}(H)$
Classical probabilistic models

Classical $\rightarrow$ deterministic noncontextual hidden variables

There are ‘hidden variables’ that determine (with certainty) which measurement outcome happens, and we only observe an average over them, according to the preparation of our physical system.
Classical probabilistic models

Classical → deterministic noncontextual hidden variables

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<td>A probabilistic model $p : V \rightarrow [0, 1]$ is classical iff</td>
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<td>$p(v) = \sum_{\lambda} q_{\lambda} p_{\lambda}(v),$</td>
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<td>where $\sum_{\lambda} q_{\lambda} = 1$, and $p_{\lambda}$ is a deterministic model for each $\lambda$.</td>
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Classical probabilistic models

Classical → deterministic noncontextual hidden variables

There are ‘hidden variables’ that determine (with certainty) which measurement outcome happens, and we only observe an average over them, according to the preparation of our physical system.

Classical model

A probabilistic model $p : V \to [0, 1]$ is classical iff

$$p(v) = \sum_\lambda q_\lambda p_\lambda(v),$$

where $\sum_\lambda q_\lambda = 1$, and $p_\lambda$ is a deterministic model for each $\lambda$.  

\[ C(H) \]
A probabilistic model $p : V \to [0, 1]$ is quantum if
\[ \exists \mathcal{H}, \rho, \{ P_v : v \in V \} \]
\[ \sum_{v \in e} P_v = 1_{\mathcal{H}} \quad \forall e \in E \]
\[ p(v) = \text{tr}(\rho P_v) \]
Quantum probabilistic models

A probabilistic model $p : V \to [0, 1]$ is quantum if

$\exists \quad \mathcal{H}, \quad \rho, \quad \{ P_v : v \in V \}$

$\sum_{v \in e} P_v = 1_{\mathcal{H}} \quad \forall e \in E$

$p(v) = \text{tr} (\rho P_v)$
State-independent contextuality

Nine measurements of four possible outcomes each.

\[ C(H) = \emptyset \text{ while } Q(H) \neq \emptyset \]
The non-orthogonality graph

“Two events are orthogonal if there exists a hyperedge that contains them both”

\[ H(V, E) \rightarrow \text{NO}(H) \]
The non-orthogonality graph

“Two events are orthogonal if there exists a hyperedge that contains them both”

\[ H(V, E) \rightarrow \text{NO}(H) \]

Example:

Contextuality scenario: \( H \)  
Non-orthogonality graph: \( \text{NO}(H) \)
NO graph and probabilistic models

\[ p \in C(H) \quad \text{iff} \quad \alpha^* (\text{NO}(H), p) = 1 \]
NO graph and probabilistic models

\[ p \in C(H) \text{ iff } \alpha^* (\text{NO}(H), p) = 1 \]

Quantum models cannot be characterised by the properties of \((\text{NO}(H), p)\)

Example: \(\exists \ H, \ H', \ p \text{ s.t.:} \)

- \(p \in Q(H)\)
- \(p \in Q_1(H') \setminus Q(H')\)
- \(\text{NO}(H) = \text{NO}(H')\)
Bell scenarios

\[ p(a_1 \ldots a_n | x_1 \ldots x_n) \]
Bell scenarios

\[ x = (x_1, \ldots, x_n) \]

\[ a = (a_1, \ldots, a_n) \]

\[ p(a_1 \cdots a_n | x_1 \cdots x_n) \rightarrow p(a | x) \]
Bell scenarios

Bell scenario $\rightarrow$ events-based hypergraph?
Bell scenarios

Bell scenario $\rightarrow$ events-based hypergraph?

Alice

Bob

$H_A$

$p_A(0|0) = 1$, $p_A(0|1) = 0$.

$H_B$

This choice of hypergraph admits signalling models.
Bell scenarios

Bell scenario → events-based hypergraph?

Alice

\[
\begin{array}{c}
H_A \\
0|0 & 1|0 \\
0|1 & 1|1
\end{array}
\]

Bob

\[
\begin{array}{c}
H_B \\
0|0 & 1|0 \\
0|1 & 1|1
\end{array}
\]
Bell scenarios

Bell scenario $\rightarrow$ events-based hypergraph?

Alice

$H_A$

Bob

$H_B$
Bell scenarios

Bell scenario → events-based hypergraph?

\[ p : (00|00) \rightarrow 1, \ (10|01) \rightarrow 1, \ (00|10) \rightarrow 1, \ (00|11) \rightarrow 1 \]
Bell scenarios

Bell scenario → events-based hypergraph?

Alice

0|0
0|1
1|0
1|1

Bob

0|0
0|1
1|0
1|1

\( H_A \)

\( H_B \)

\( p : \ (00|00) \rightarrow 1, \ (10|01) \rightarrow 1, \ (00|10) \rightarrow 1, \ (00|11) \rightarrow 1 \)

\( p_A(0|00) = 1, \ p_A(0|01) = 0. \)

This choice of hypegraph admits signalling models
Bell scenarios

Correlated measurements

- temporal order of the parties. e.g. $A \rightarrow B$
- a choice of measurement for the first party, e.g. $x$.
- a function $y = f(a)$ for the second party, that determines its measurement input as a function of the previous party’s outcome.

Example:

$(A \rightarrow B, x = 0, y = a)$

Correlated measurement with outcomes $\{(00|00), (01|00), (10|01), (11|01)\}$. 
Bell scenarios

Events-based hypergraph: $B_{n,m,d}$

- Vertices: events
- Edges: correlated measurements

$G(B_{n,m,d}) = NS(n,m,d)$

$C(B_{n,m,d}) = C(n,m,d)$

$Q(B_{n,m,d}) = Q(n,m,d)$

$\Pi_{ab|xy} = \Pi_a|x \Pi_b|y$ with $[\Pi_a|x, \Pi_b|y] = 0$

$B_{n,m,d}$ may be computed via the "Foulis-Randall" product of the local hypergraphs.
Bell scenarios

Events-based hypergraph: $\mathcal{B}_{n,m,d}$
- Vertices: events
- Edges: correlated measurements

- $G(\mathcal{B}_{n,m,d}) = \mathcal{N}(n, m, d)$
- $\mathcal{C}(\mathcal{B}_{n,m,d}) = \mathcal{C}(n, m, d)$
- $Q(\mathcal{B}_{n,m,d}) = Q(n, m, d)$  \(\rightarrow\)  $\Pi_{ab|xy} = \Pi_{a|x} \Pi_{b|y}$ with $[\Pi_{a|x}, \Pi_{b|y}] = 0$
Bell scenarios

Events-based hypergraph: $\mathcal{B}_{n,m,d}$
- Vertices: events
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- $G(\mathcal{B}_{n,m,d}) = N S(n, m, d)$
- $C(\mathcal{B}_{n,m,d}) = C(n, m, d)$
- $Q(\mathcal{B}_{n,m,d}) = Q(n, m, d)$ $\quad \rightarrow \quad \Pi_{ab|xy} = \Pi_{a|x} \Pi_{b|y}$ with $[\Pi_{a|x}, \Pi_{b|y}] = 0$

- $\mathcal{B}_{n,m,d}$ may be computed via the “Foulis-Randall” product of the local hypergraphs.
Relation to Compatible-observables scenarios

**Scenario :** \((X, O, \mathcal{M})\)

- \(X\): observables
- \(O\): outcomes
- \(\mathcal{M}\): measurement cover \(\rightarrow\) sets of compatible measurements
  
  \(C \in \mathcal{M}\) measurement context
Relation to Compatible-observables scenarios

Scenario : \((X, O, M)\)

- \(X\): observables
- \(O\): outcomes
- \(M\): measurement cover \(\rightarrow\) sets of compatible measurements
  \[C \in M\] measurement context

KCBS:

\[X = \{A_i : 1 \leq i \leq 5\}\]

\[O = \{-1, 1\}\]

\[M = \{\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}, \{A_4, A_5\}, \{A_5, A_1\}\}\]
Relation to Compatible-observables scenarios

\[(X, O, M) \rightarrow H[X]\]

- **Vertices:**

- **Hyperedges:**

  - Measurement protocols
    - Choose and measure an observable \(A\).
    - Depending on the outcome, choose a compatible observable \(A'\).
    - Measure \(A'\), ...
Relation to Compatible-observables scenarios

\[(X, O, \mathcal{M}) \rightarrow H[X]\]

- Vertices: \((s, C) : C \in \mathcal{M}, s \in O^C\)

- KCBS: \((a_i a_{i+1}|A_i A_{i+1})\)

- Hyperedges:
Relation to Compatible-observables scenarios

$$(X, O, M) \rightarrow H[X]$$

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  KCBS: $(a_i a_{i+1} | A_i A_{i+1})$

- Hyperedges: Measurement protocols

  KCBS:

  - Choose and measure an observable ($A$).
  - Depending on the outcome, choose a compatible observable ($A'$).
  - Measure ($A'$), …

  $$f : \{1, -1\} \rightarrow \{A_{k-1}, A_{k+1}\}$$
Contextuality bundles

Tool to demonstrate the obstruction to a NCHV model
Contextuality bundles

Tool to demonstrate the obstruction to a NCHV model

Probability table:

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Contextuality bundles

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- **Local section**: assignment of ‘possible’ values for a given context.
- **Global section**: collection of ‘compatible’ local sections
Contextuality bundles

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No local section can be extended to a global one

→ Strong Contextuality
Contextuality bundles

A quantum example

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Some local sections cannot be extended to a global one → Logical Contextuality
Contextuality bundles

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<tr>
<td>11</td>
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</tbody>
</table>

Some local sections cannot be extended to a global one

→ Logical Contextuality
Summary of today

- Scenarios from operational equivalences
  - Define sets of probabilistic models
  - Graph theoretical advantages
  - Bell scenarios
  - Compatible-observables scenarios (S. Abramsky and A. Brandenburger, New J. Phys. 13(11), 113036 (2011).)

- Contextuality bundles
Closing remarks

- Kochen-Specker contextuality, with focus on graph theory

- Hidden variable models, and quantum violations

- Contextuality scenarios: ‘observables with compatibility relations’, or ‘events with operational equivalences’.

- Types of probabilistic models $\rightarrow$ graph theory
  - CSW: contextuality
  - ALFS: set membership

- Bell scenarios as contextuality ones
Summary

- **Kochen-Specker contextuality**

- **KCBS example**

- **State-independent contextuality**
  A. Peres, Phys. Lett. A 151, 107-8 (1990).*

- **Inequalities from hypergraphs: CSW approach**
  - KCBS, CHSH Bell scenario, Limitations: $l_{3322}$
  *A. Cabello, S. Severini, A. Winter, arXiv:1010.2163*

- **Scenarios from operational equivalences**
  - Graph theoretical advantages, Bell and Compatible-observables scenarios

- **Contextuality bundles**
“Dilation”, a.k.a. “The Church of the larger Hilbert space”:

Let $p(a|x)$ be a probabilistic model in a contextuality scenario. Let us assume that we have a realisation of it in terms of POVMs (i.e. generalised measurements); that is, a Hilbert space $\mathcal{H}$, positive semidefinite matrices $M_{a|x}$ s.t. $\sum_a M_{a|x} = 1_{\mathcal{H}}$ and a quantum state $\rho$, s.t. $p(a|x) = \text{tr} \left\{ M_{a|x} \rho \right\}$.

A dilation of this model is a realisation of the correlations in terms of projective measurements, i.e. a Hilbert space $\mathcal{H}'$ (possibly of larger dimension than $\mathcal{H}$), projectors $\Pi_{a|x}$ s.t. $\sum_a \Pi_{a|x} = 1_{\mathcal{H}'}$ and a quantum state $\rho'$, s.t. $p(a|x) = \text{tr} \left\{ \Pi_{a|x} \rho' \right\}$.

Colloquially, the POVM operators $M_{a|x}$ are dilated into projection operators $\Pi_{a|x}$.

When can we find a dilation of a POVM model?

The mathematical counterpart of this question was addressed by Naimark and Stinespring in the context of C*-algebras. See e.g.,


Comments on Dilation

Contextuality scenarios:

It is not always possible to dilate a POVM realisation of a probabilistic model, such that the dilated projectors satisfy the same compatibility relations as the original POVM elements.


Bell scenarios:

Any POVM realisation of Bell correlations has an equivalent realisation in terms or projective measurements.


V. Paulsen, Lecture notes on “Entanglement and Non-Locality” (Sec. 9), available at http://www.math.uwaterloo.ca/~vpaulsen/ (does not use C*-algebras)