Kochen-Specker contextuality: a hypergraph approach with operational equivalences

Ana Belén Sainz,
A. Acín , T. Fritz, A. Leverrier

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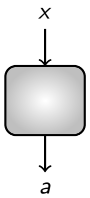
Outline

- Hypergraph framework
- Bell scenarios
- Compatible-observables scenarios
- Graph theory advantage
- Quantum from principles

Contextuality scenarios

- Set of measurements
- Set of outcomes



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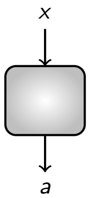
- Operational equivalences $\rightarrow$ identify outcomes of different measurements: same probability



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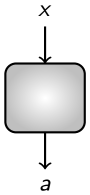

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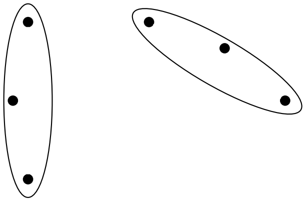

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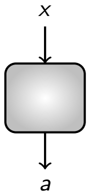

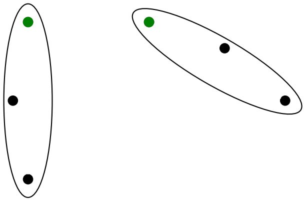

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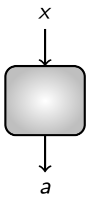

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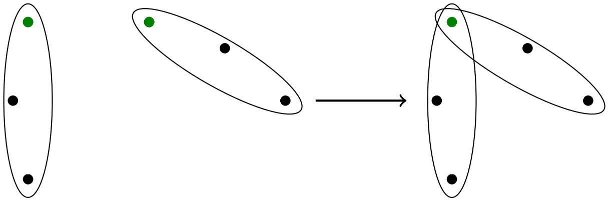

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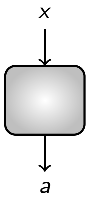
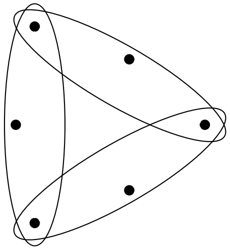
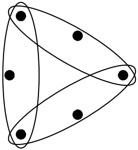


Hypergraph:

- Vertices $\rightarrow$ events – measurement outcome
- Hyperedges $\rightarrow$ complete measurements – set of outcomes



Contextuality Scenarios



- **Probabilistic Model**: $\mathcal{G}(H)$
  $p : V \to [0,1]$, such that $\sum_{v \in e} p(v) = 1$, for each $e \in E$.

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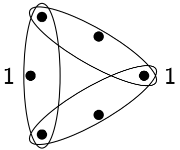

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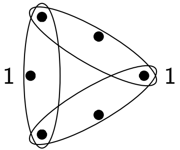

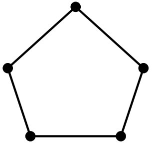

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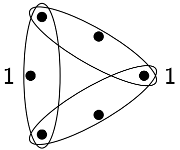

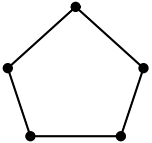

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- **Classical models**: $\mathcal{C}(H)$
  $p\,(v) = \sum_{\lambda} q_{\lambda} p_{\lambda}(v)$
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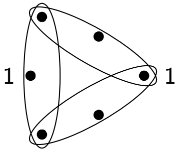
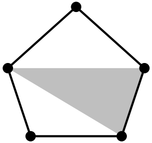
Contextuality Scenarios





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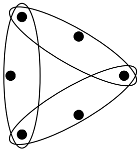

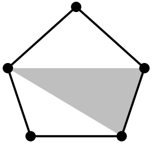

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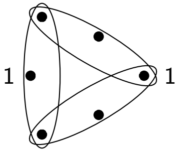

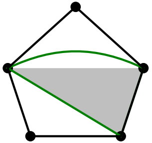

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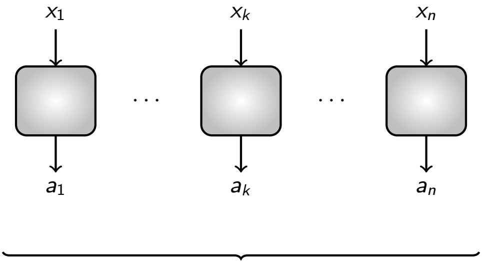
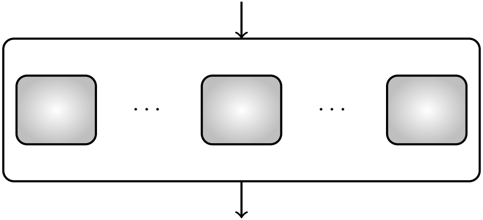
Bell scenarios



$$p\left(a_1 \ldots a_n \middle| x_1 \ldots x_n\right)$$

Bell scenarios

$$x = (x_1, \dots, x_n)$$



$$a = (a_1, \dots, a_n)$$

$$p\left(a_1 \dots a_n | x_1 \dots x_n\right) \quad \rightarrow \quad p\left(a|x\right)$$

Bell scenarios

$$(n, m, d) \quad \rightarrow \quad \mathcal{B}_{n,m,d} \quad ?$$

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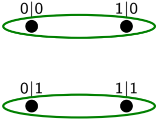
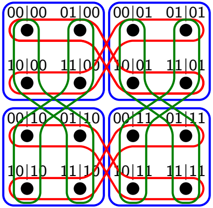
Alice



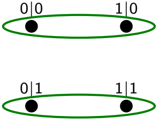
H_A



Foulis and Randall:

$$H_{AB} = H_A \otimes_{\mathrm{FR}} H_B$$

Bob



H_B

Bell Scenarios

Product Scenarios

Alice: H_A ,   Bob: H_B $\longrightarrow$ Joint scenario: $H_A \otimes H_B$

Foulis-Randall product

$$\mathcal{B}_{n,m,d} = \otimes_{k=1}^{n} H_k = \mathcal{B}_{1,m,d}^{\otimes n}$$

- $\mathcal{G}(\mathcal{B}_{n,m,d}) = \mathcal{NS}(n, m, d)$
- $\mathcal{C}(\mathcal{B}_{n,m,d}) = \mathcal{C}(n, m, d)$
- $\mathcal{Q}(\mathcal{B}_{n,m,d}) = \mathcal{Q}(n, m, d) \quad \longrightarrow \quad P_{a_1 \ldots a_n | x_1 \ldots x_n} = \prod_j P_{a_j | x_j}$

Compatible-observables scenarios[1]

Scenario : $(X, O, \mathcal{M})$

- X: observables
- O: outcomes
- $\mathcal{M}$: measurement cover $\rightarrow$ sets of compatible measurements
  $C \in \mathcal{M}$ measurement context

[1] Abramsky Brandenburger, NJP 13 (2011), no. 11, 113036.

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Bell scenario (2,2,2):

$$\{A_1, B_1\}, \quad \{A_1, B_2\}, \quad \{A_2, B_1\}, \quad \{A_2, B_2\}$$

$$\mathcal{M} = \{\{A_1, B_1\}, \{A_1, B_2\}, \{A_2, B_1\}, \{A_2, B_2\}\}$$

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Compatible-observables scenarios

$$(X, O, \mathcal{M}) \quad \longrightarrow \quad H[X]$$

- Vertices:

- Hyperedges:

Compatible-observables scenarios

$$(X, O, \mathcal{M}) \quad \longrightarrow \quad H[X]$$

- Vertices: $(C, s) : C \in \mathcal{M}, s \in O^C$

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- Hyperedges: Measurement protocols

- Choose and measure an observable (A).

- Depending on the outcome,
choose a compatible observable (A').

- Measure (A'), . . .

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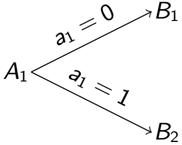

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- Vertices: $(C, s) : C \in \mathcal{M}, s \in O^C$
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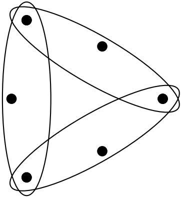
Empirical model $P_C(s)$ $\Leftrightarrow$ **Probabilistic model** $p(C, s)$

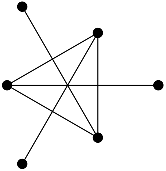
Graph Theory advantage

Contextuality scenario: H



$\longrightarrow$

Non-orthogonality graph: NO(H)



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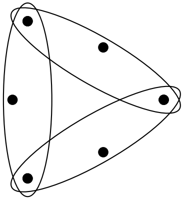

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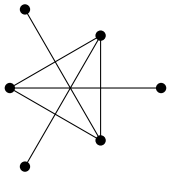


$$p \in \mathcal{C}(H) \quad \text{iff} \quad \alpha^* \left(\mathrm{NO}(H), p\right) = 1$$

Quantum models

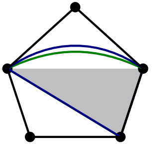
Quantum models cannot be characterised by the properties of $(\mathrm{NO}(H), p)$

Example: $\exists \quad H, \quad H', \quad p \quad$ st:

- $p \in \mathcal{Q}(H)$
- $p \in \mathcal{Q}_1(H') \setminus \mathcal{Q}(H')$
- $\mathrm{NO}(H) = \mathrm{NO}(H')$

Almost quantum models

"Almost" quantum: $\mathcal{Q}^{1+\mathrm{AB}}$ [4] , $\mathcal{Q}_1$ [5] , $\tilde{\mathcal{Q}}$ [6]



Attempts to characterise quantum $\longrightarrow$ Almost quantum

$\mathcal{Q}_1$

$\exists \quad \mathcal{H}, \quad \rho, \quad \{P_v : v \in V\}$

$\sum_{v \in e} P_v \leq \mathbb{1}_{\mathcal{H}} \qquad \forall\, e \in E$ $\qquad \mathcal{Q}_1(H)$

$p(v) = \mathrm{tr}(\rho P_v)$

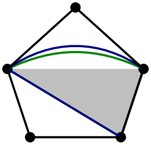
[4] M. Navascués, S. Pironio and A. Acín, NJP 10 (7), 073013 (2008)

[5] A. Acín, T. Fritz, A. Leverrier and ABS, CMP 334(2), 533-628 (2015)

[6] M. Navascués, Y. Guryanova, M. Hoban and A. Acín, Nat Comm 6, 6288 (2015)

Almost quantum models

"Almost" quantum: $\mathcal{Q}^{1+\mathrm{AB}}$, $\mathcal{Q}_1$, $\tilde{\mathcal{Q}}$



Attempts to characterise quantum $\longrightarrow$ Almost quantum

$p \in \mathcal{Q}_1(H)$ if and only if $\vartheta(\mathrm{NO}(H), p) = 1$

Quantum from principles: Consistent Exclusivity

CE: Contextuality[2]

Exclusiveness : if $\exists\, e \in E$ with $\{u, v\} \subset e \quad \Rightarrow \quad u \perp v$

Constraint: $\sum_{v \in S} p(v) \leq 1$, S set of pairwise exclusive events

$p(a|x) \in$ CE[1]

[2] A. Cabello, S. Severini and A. Winter, arXiv:1010.2163 (2010) – A. Acín, T. Fritz, A. Leverrier, ABS, CMP 334, 533-628 (2015)– Joe Henson, arXiv:1210.5978 (2012)

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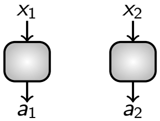

$p(a|x) \in \text{CE}^1$

$$p(a_1 a_2|x_1 x_2) := p(a_1|x_1)\, p(a_2|x_2) \in \text{CE}^1$$

$$p(a|x) \in \text{CE}^2$$

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Quantum from principles: Consistent Exclusivity

Consistent Exclusivity: $\sum_{v \in S} p(v) \leq 1$

S independent set on $\mathrm{NO}(H)$

CE^1 $p \in \mathscr{CE}^1(H)$ if and only if $\alpha(\mathrm{NO}(H), p) = 1$

CE^k $p \in \mathscr{CE}^k(H)$ if and only if $\alpha(\mathrm{NO}(H)^{\boxtimes k}, p^{\otimes k}) = 1$

CE$^\infty$ $p \in \mathscr{CE}^\infty(H)$ if and only if $\Theta(\mathrm{NO}(H), p) = 1$

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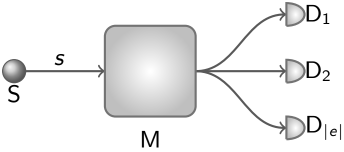
$$\alpha \leq \Theta \leq \vartheta \quad \Rightarrow \quad \mathcal{Q}_1(H) \subseteq \mathscr{CE}^\infty(H)$$

Macroscopic Non-Contextuality

Micro scenario



$p(v)$

Macroscopic Non-Contextuality

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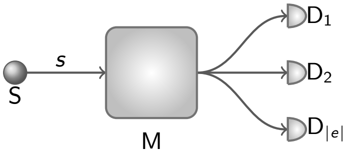
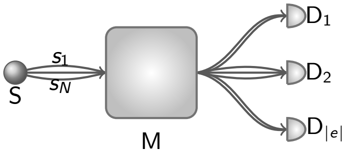


Macro scenario



$p(v)$

$\mathcal{P}_e\left(\{I^v\}_{v\in e}\right)$

Macroscopic Non-Contextuality

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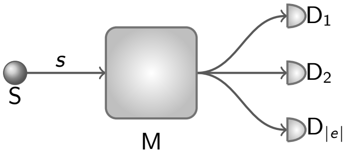

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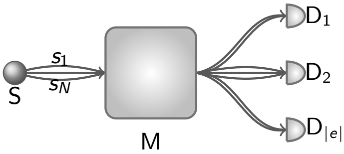


$p(v)$

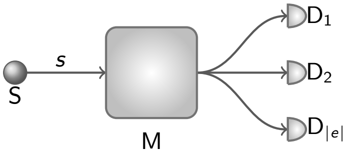
$\mathcal{P}_e\left(\{I^v\}_{v\in e}\right)$

Macroscopic Non-Contextuality:

$p(v)$ **satisfies MNC if** $(N\to\infty)$ $\mathcal{P}_e\left(\{I^v\}_{v\in e}\right)$ **is noncontextual**

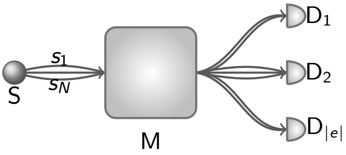
Macroscopic Non-Contextuality

Micro scenario



$p(v)$

Macro scenario



$\mathcal{P}_e\left(\{I^v\}_{v\in e}\right)$

Macroscopic Non-Contextuality:

$p(v)$ satisfies MNC if $(N \to \infty)$ $\mathcal{P}_e\left(\{I^v\}_{v\in e}\right)$ is noncontextual

$p(v)$ satisfies MNC iff $p \in \mathcal{Q}_1$

Conclusions and open problems

- Framework to unify Nonlocality and Contextuality
- Useful connection to Graph Theory

- Temporal scenarios – sequences of measurements
- New principles to constrain probabilistic models

  $\longrightarrow$ Distinguish quantum from almost quantum

- Applications

Thanks!

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- Applications

A. Acín, T. Fritz, A. Leverrier and ABS, Comm. Math. Phys. 334(2), 533-628 (2015).

T. Fritz, A. Leverrier and ABS, EPTCS 171, pp. 63-70 (2014).