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that's my first name ☺

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Overview
1. classical causality recap
2. Quantum examples
3. Frameworks (Process matrices, causal boxes)
4. insights for foundations
5. Applications

CAUSALITY II : Lecture Notes

I. Recap : Classical causality

Observed correlations $\rightleftarrows$ Cause-effect relationships

E.g. P_{XYZ}

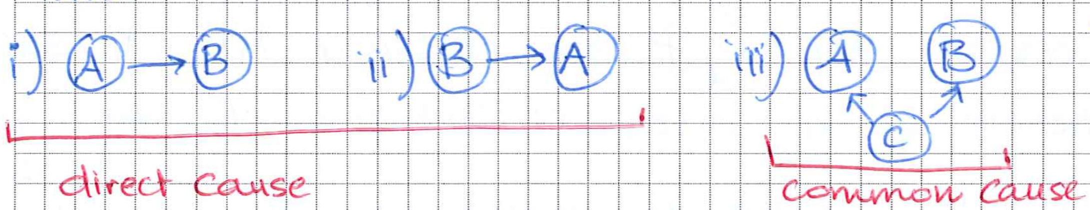
X, Y, Z : Some observed random variables (RVs)

DAGs with observed nodes: X, Y, Z and some unobserved nodes: Λ

- When all unobserved nodes are also classical RVs, ∃ well-established framework: classical causal models (Pearl 2000)
- Cornerstone of this framework: Reichenbach's principle.

→ Reichenbach's principle

If 2 RVs A and B are correlated, i.e $P_{AB} \neq P_A P_B$, then



or combination of direct & common cause

$\Rightarrow P_{AB|C} = P_{A|C} P_{B|C}$ (conditional independence CI)

- CI implies that 'correlations between A & B disappear once C becomes known'

→ More generally, for an n-node DAG $\mathcal{G}$, associated joint distribution $P_{X_1,\dots,X_n}$ over the nodes satisfies the Causal Markov Condition :

$$P_{X_1,\dots,X_n} = \prod_{i=1}^{n} P_{X_i|par(X_i)}$$

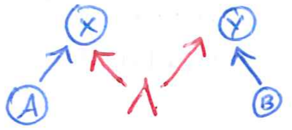
$par(X_i)$: set of all parents of X_i in $\mathcal{G}$

condition for compatibility with causal structure (see Mirjam's lecture)

II. Quantum Causality

→ Bell's Theorem

$\mathcal{G}_{Bell}$:



Causal Markov condition:

$$P_{ABXY\lambda} = P_\lambda P_A P_B P_{X|A\lambda} P_{Y|B\lambda}$$

$$\Rightarrow P_{XY\lambda|AB} = P_\lambda P_{X|A\lambda} P_{Y|B\lambda}$$

$$\Rightarrow \boxed{P_{XY|AB} = \sum_\lambda P_\lambda P_{X|A\lambda} P_{Y|B\lambda}}$$

→ Local causality condition! (LC)

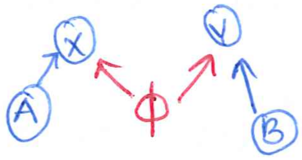
Conditioning on λ makes X, Y correl. disappear

(assumptions of no-signalling and free-choice evident from DAG since A & B are independent & not B → X, A → Y arrows)

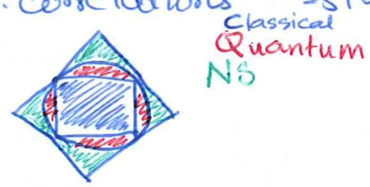
⇒ Since LC ⇒ Bell inequalities (BI) and LC must be satisfied by all correlations $P_{XY|AB}$ arising from the classical causal structure $\mathcal{G}_{Bell}$, BIs are simply constraints on observed correlations in this classical causal structure.

- But QM violates BI ⇒ for some quantum correlations, $P_{XY|AB}$, ∄ no classical RV λ such that conditioning on it makes distribution factorise as in LC.

Quantum $P_{XY|AB}$ generated by measuring q. state ϕ with settings A, B to get out-comes X, Y



⇒ Q. correlations stronger than classical.



Wood & Spekkens: Classical causal models cannot satisfactorily explain quantum correlations.
without fine-tuning (retrocausality, superdeterminism etc.)

∴ $\boxed{\text{Notion of causality different in non-classical theories}}$

⇒ Existing framework (Pearl 2000) does not work for these!

→ Superpositions of causal and temporal orders

- Typically QM assumes fixed order of operations / fixed background space-time structure. What if causal / temporal orders are in quantum superpositions?
  E.g. Quantum + gravity scenario (superpos. of gravitating mass)
- Even without QG, can have indefinite temporal order
  ⇒ cases that can't be written as quantum circuits
  operations in pre-defined time order
  E.g. quantum switch (QS) (Chiribella 2013)

★ QS

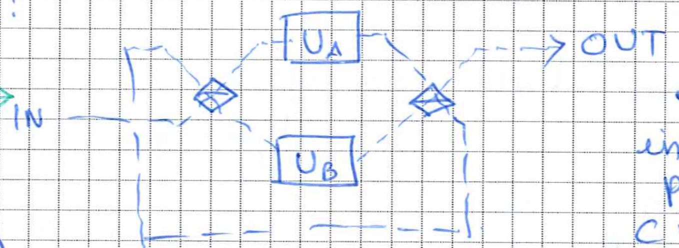
2 black-box unitaries U_A, U_B applied in controlled superpos. of orders. Only one query each!

$$(\alpha|H\rangle_C + \beta|V\rangle_C) \otimes |\psi\rangle_T \xrightarrow{QS} \alpha|H\rangle_C (U_B U_A|\psi\rangle)_T + \beta|V\rangle_C (U_A U_B|\psi\rangle)_T$$

$(|\alpha|^2 + |\beta|^2 = 1)$

Cannot be represented as a q. circuit without querying either U_A or U_B atleast twice!

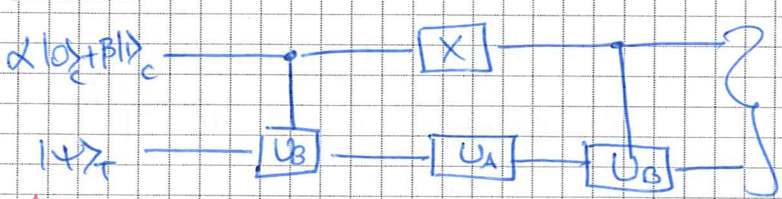
Realisation:



(Procopio 2015
Rubino 2017)

- control (C) & target (T) in 2 d.o.fs of same photon.
  C: polarization
  T: angular momentum modes
- $|H\rangle$: transmitted by PBS
  $|V\rangle$: reflected by PBS

Circuit:



$$\alpha|0\rangle_C\, U_B U_A|\psi\rangle_T + \beta|1\rangle_C\, U_A U_B|\psi\rangle_T.$$

QS provides ↑ Computational advantage over fixed orders in certain tasks (Araujo 2014)

- Also ∃ more exotic examples in theory for which we don't know of any physical implementation. (more later)

III. Main approaches for studying causality

(1) Classical $\xrightarrow{\text{generalise}}$ Quantum

E.g. JM. Allen et al 2017 (Quantum Reichenbach)

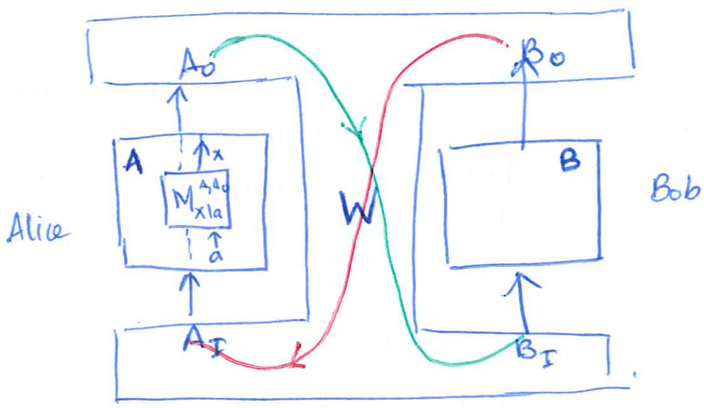
(2) More general notions of causality $\xrightarrow[\text{down}]{\text{narrow}}$ Quantum.

E.g. process matrices (PM), causal boxes (CB) (many others, but we focus on these).

- Here, we focus on (2).
- Comparing Qs in PM & CB frameworks gives interesting foundational insights! (See: my masters thesis)

IV. Process Matrices (Oreshkov et. al 2012)

- Local quantum labs but no global space-time structure outside labs.



$W^{A<B}$
$W^{B<A}$

Local input, output

- $\mathcal{H}_{A_I}$, $\mathcal{H}_{A_O}$: input, output Hilbert spaces of dim d_{A_I}, d_{A_O} for Alice.

A_I, A_O : set of density matrices on $\mathcal{H}_{A_I}$, $\mathcal{H}_{A_O}$.

- local quantum operations:

Quantum instrument $\mathcal{J}^A_a = \{M^A_{x|a}\}^m_{x=}$

describes measurements & other operations /transformations. Keeps track of C & Q inputs & outputs.

C-J representation: $M^A_{x|a} : A_I \to A_O$ (channel) $\Leftrightarrow$ $M^{A_I A_O}_{x|a} \in A_I \otimes A_O$ (state), ≥ 0 (positive operators).

- Global description

- No fixed metric / space-time outside labs.
- Causal connections encoded in process matrix W, which is outside the control of agents.

- Operational predictions: generalised 'Born Rule'

N local labs: $A_1 \ldots A_N$, $W \in A_I^1 \otimes A_O^1 \otimes \cdots \otimes A_I^N \otimes A_O^N$ (Hermitian operator)

Probability that N agents observe outcomes $x_i \in X$ for settings $a_i \in A_i$:

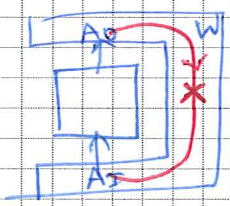
$$\boxed{P_{X_1 \ldots X_N | A_1 \ldots A_N}(x_1, \ldots, x_N | a_1, \ldots, a_N) = \mathrm{Tr}\left[\left(M^{A_I^1 A_O^1}_{x_1|a_1} \otimes \cdots \otimes M^{A_I^N A_O^N}_{x_N|a_N}\right)^T \cdot W\right]}$$

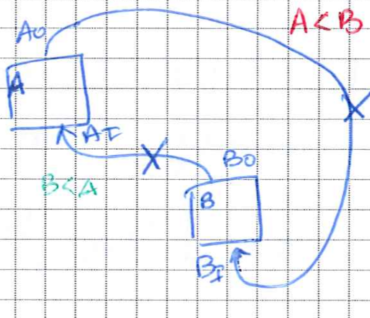
CJ reps of quantum instruments (local)

plays the role of a 'generalised' state

★ Valid PMs defined as those that give valid positive, normalised probabilities $\forall$ possible operations of N agents $\Rightarrow$ $W \geq 0$ (among other constraints)

★ This rules out causal loops!





forbidden PMs!

→ But can have certain superpositions of the 2 orders! (e.g. QS)

→ Causally separable vs non-separable PM

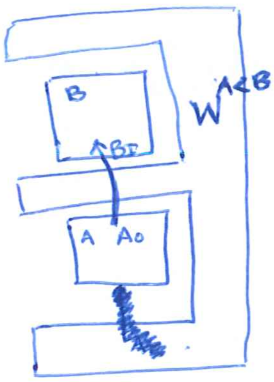
Theory & device ~~in~~dependent certificate of indefinite ~~non-classicality~~ ~~of~~ causal/temporal orders (analogy: Entanglement witness).

.. First define causally ordered PMs ~~class~~

We say that a 2 party PM $W \in A_I \otimes A_O \otimes B_I \otimes B_O$ is compatible with the causal order $A<B$ if

$\left(\text{for } {}_xW := \frac{\mathbb{1}_x}{d_x} \otimes \mathrm{tr}_x W\right)$, $W = {}_{B_O}W$ and ${}_{B_I B_O}W = {}_{A_O B_I B_O}W$

NOTE: In this case, A has trivial input & B has trivial output (i.e dimension 1)



• Causal separability$^{(cs)}$ (bipartite) :=

$W \in A_I \otimes A_O \otimes B_I \otimes B_O$ is C.S. if it can be written as follows for some $W^{A<B}$, $W^{B<A}$, q

$$W^{sep} = q\, W^{A<B} + (1-q)\, W^{B<A}, \quad q \in [0,1]$$

causally ordered PMs.

else, it is causally non-separable!

• Causal Witness (Araujo et al. 2015): A hermitian operator S is called a causal witness if $\mathrm{tr}[S W^{sep}] \geq 0$ for every causally separable PM W^{sep}.

- set of W^{sep} closed & convex

$\Rightarrow$ for every causally non-sep. W^{ns}, $\exists\, S_{W^{ns}}$ such that $\mathrm{tr}[S_{W_{ns}} W^{ns}] < 0$.

- Can be measured in lab!

→ Causal inequalities

Device and theory independent certificate of ~~noncausality of~~ indefinite causal/temporal orders (analogy: Bell inequalities)

- First define causally ordered distributions.

$P(xy|ab)$ is compatible with the order $A<B$ if

$$\forall\ a, b, b', x, \quad p^{A<B}(x|ab) = p^{A<B}(x|ab')$$

if $A<B$, then B can't signal to A

'causal' correlations (bipartite): $P(xy|ab)$ is causal if $\exists\ p^{A<B}, p^{B<A}, q$ such that:

$$\boxed{P(xy|ab) = q\,p^{A<B}(xy|ab) + (1-q)\,p^{B<A}(xy|ab)} \quad (*)$$

(analog of local causality condition for CHSH)

- Note: no-signalling distributions are compatible with both $A<B$ & $B<A$.

- Form a 'causal' polytope analogous to Bell-local polytope.

• Guess your neighbour's input (GYNI) game (analog: CHSH game)
  - 2 parties, A & B, each can let a system enter/exit their labs only once!
  - Pick uniform random inputs $a, b \Rightarrow p(ab) = 1/4$

  Have to output x, y, guess of other party's input.

  Win if $x = b$, $y = a$.

$$\boxed{P_{GYNI} = P(x=b, y=a) \leq \frac{1}{2} \text{ is a causal ineq.}}$$

Obeyed by all 'causal' distributions $P(xy|ab)$

Why?: • For $A<B$: $P(y=a) = 1$ possible (A sends a to B) but B can't signal to A $\Rightarrow P(x=b) = 1/2$ (A guesses at random)

$\therefore P(x=b, y=a) \leq 1/2$. Similarly for $B<A$.

• Convex comb. of the 2 can't increase bound!

$\exists$ W that violate this causal inequality!

$$W^* = \frac{1}{4}\left[\mathbb{1}^{\otimes 4} + \frac{Z^{A_I} Z^{A_0} Z^{B_I} \mathbb{1}^{B_0} + Z^{A_I} \mathbb{1}^{A_0} X^{B_I} X^{B_0}}{\sqrt{2}} \right]$$

for suitable local measurements yields

$P_{GYNI} \approx 0.5335 > \frac{1}{2}$

$\mathbb{1}$: identity
Z, X : Pauli X, Z.

- Analog of Tsirelson's bound unknown.

★ One-way communication is crucial! (Trivial with 2 way)

★ Currently not known if W_A^* is physically implementable (or any other W that violates causal ineqs)

★ Possible to violate causal ineq. using quantum fields delocalized in space and time, even within a fixed Minkowski space-time! (Ho et. al 2018)

- This can't be embedded in PM framework.

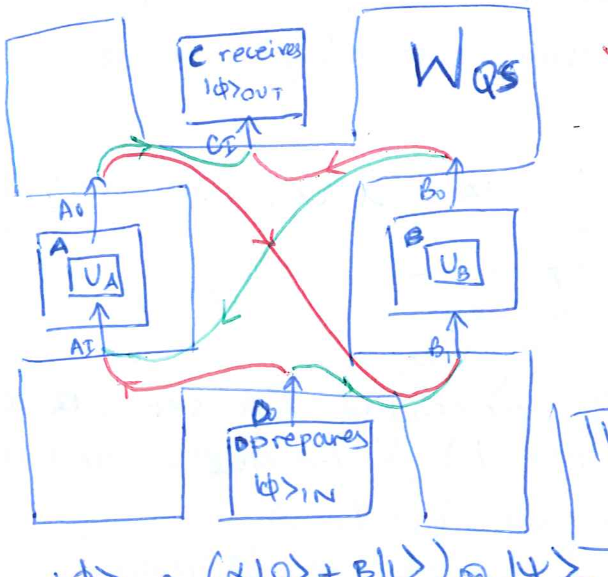
In general, No tensor products in QFT for composite Hilbert spaces

★ PMs correspond to linear 2-time states (Silva et. al 2017) arXiv: 1701.08638 (NJP)

→ QS as a process matrix

$|\psi\rangle_{OUT} = \alpha|0\rangle_c U_B U_A |\psi\rangle_T + \beta|1\rangle_c U_A U_B |\psi\rangle_T$

- D has trivial input, C has trivial output



$|\psi\rangle_{IN} = (\alpha|0\rangle_c + \beta|1\rangle_c) \otimes |\psi\rangle_T$

if control is $|0\rangle_c$, follow red path: $D < A < B < C$

if control is $|1\rangle_c$, follow green path: $D < B < A < C$.

$W_{QS} = |W_{QS}\rangle\langle W_{QS}|$

id. channel

★ $|W_{QS}\rangle = \alpha|00\rangle^{C_0^c D_I^c} |\mathbb{1}\rangle\rangle^{C_0^t A_I} |\mathbb{1}\rangle\rangle^{A_0 B_I} |\mathbb{1}\rangle\rangle^{B_0 D_I^t}$
$+ \beta|11\rangle^{C_0^c D_I^c} |\mathbb{1}\rangle\rangle^{C_0^t B_I} |\mathbb{1}\rangle\rangle^{B_0 A_I} |\mathbb{1}\rangle\rangle^{A_0 D_I^t}$

control target

★ W_{QS} is causally non-separable but does not violate causal ineqs! (analogy: Bound entanglement)

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V. Causal Boxes (Portmann et. al. 2017)

- A causal box (CB) can be seen as a ~~m~~ CPTP map from states on an input wire to those on an output wire that obeys causality.
- CBs are closed under composition.

How is it different from a usual quantum channel?

- Different message / wire state spaces.

- Single message space: a message is a pair (v, t)
$v \in V$ (classical/quantum message), $t \in T$ (space-time location/ some order/position info.)
T is a countable partially ordered set.

State space of messages: Hilbert space with basis $\{|v,t\rangle\}_{v\in V, t \in T}$

- For finite V, infinite T, $\mathcal{H} = \mathbb{C}^{|V|} \otimes \ell^2(T)$

Sequence space with bounded 2-norm

e.g. $|t\rangle$ can be a vector with '1' corresponding to t & '0' elsewhere

- Wires: Can carry multiple messages of fixed dimension d, also a superposition of different numbers of messages of dim d.

$\Rightarrow$ Symmetric (bosonic) Fock space

contains all ordering info — to model the superpos. of different # of messages

for $\mathcal{H} = \mathbb{C}^d \otimes \ell^2(T)$ (qudit with location info)

Fock space: $\mathcal{F}(\mathbb{C}^d \otimes \ell^2(T)) := \bigoplus_{n=0}^{\infty} \vee^n (\mathbb{C}^d \otimes \ell^2(T))$

$\vee^n \mathcal{H}$: symmetric subspace of $\mathcal{H}^{\otimes n}$

$\mathcal{H}^{\otimes 0}$ is 1D space of vacuum state $|\Omega\rangle$

Examples: 1) $\frac{1}{\sqrt{2}}(|0,t\rangle + |1,t\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ sent at time t

2) $\frac{1}{\sqrt{2}}(|0,t_1\rangle + |0,t_2\rangle) \rightarrow |0\rangle$ sent at superpositions of 2 times t_1 & t_2.

3) $\underbrace{|0,t_1\rangle \otimes |1,t_2\rangle + |1,t_2\rangle \otimes |0,t_1\rangle}_{\text{Symmetric state}} \rightarrow |0\rangle$ sent at t_1 and $|1\rangle$ sent at t_2

$\rightarrow$ Types of wires: dim. of messages they carry.

→ Cuts and causality

Cut := A cut of a partially ordered set T is any $C \subseteq T$ such that $C = \bigcup_{t \in C} T^{\leq t}$

where $T^{\leq t} := \{p \in T : p \leq t\}$.

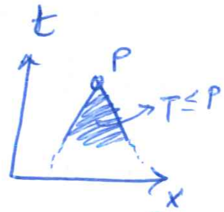
$\mathcal{C}(T)$: set of all cuts.

Bounded cut := A cut C is bounded if $\exists\ t \in T$ such that $C \subseteq T^{\leq t}$

$\tilde{\mathcal{C}}(T)$: set of all bounded cuts.

E.g. Minkowski space (rest of talk) M.

$T^{\leq P}$: inclusive past of space-time point P



- Cuts are space-time regions such that if a point p belongs to this region, so does its entire causal past (past light cone).

all cuts are bounded in M! always exists P in distant future of everything.

Causality: "Any output of a CB can only depend on inputs produced in its causal past".

Formally, For T being time (total order), $\exists\ X : T \to T$ such that outputs upto time t can be computed from inputs upto time $X(t) < t$.

- For T being space-time (partial order), define X on cuts.

Causality function (X) : $X : \mathcal{C}(T) \to \mathcal{C}(T)$ & satisfies some consistency conditions. (see original paper/my thesis for details)

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- Formal definition of a causal box: ↗ special case: finite CB. See paper for general definition

A (d_X, d_Y) CB $\hat{\Phi}$ is a system with input wire X & output wire Y of dimensions d_X & d_Y defined by a CPTP map:

$$\Phi : \mathcal{L}(\mathcal{F}_X^T) \longrightarrow \mathcal{L}(\mathcal{F}_Y^T)$$

↖ set of "density matrices" on the Fock space..

satisfying causality: $\exists$ causality function $\chi : \mathcal{C}(T) \to \mathcal{C}(T)$ such that

$$\boxed{\Phi^C := tr_{T \setminus C} \circ \Phi = \Phi^C \circ tr_{T \setminus \chi(C)}}$$

↳ tells us that every output of the causal box in the cut C ~~of~~ is independent of inputs outside of $\chi(C)$.

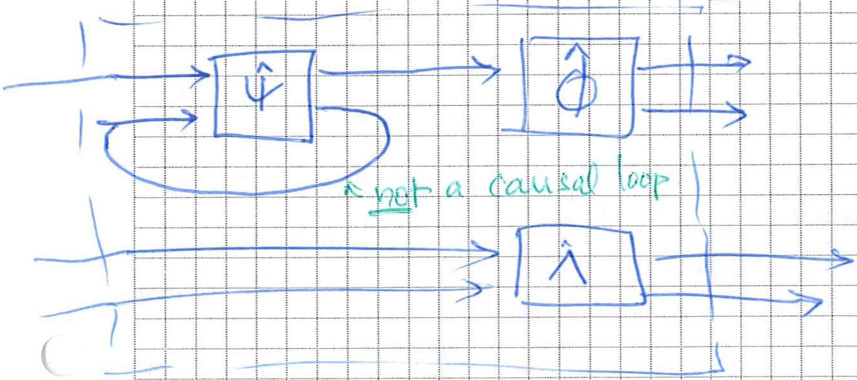
Remark: Enough to consider single wires (w.l.o.g) since ~~every~~ wires of dim $d_1, .. d_n$ can be combined into single wire of $d = d_1 + d_2 .. + d_n$ (& vice versa). (non-trivial: see paper)

- ~~Composit~~ • Representations

Since a CB is a CPTP map, it has a CJ, Stinespring representations. Also has a 'sequence' representation. (Causal unravelling of the map: steps occurring within each time (space-time slice))

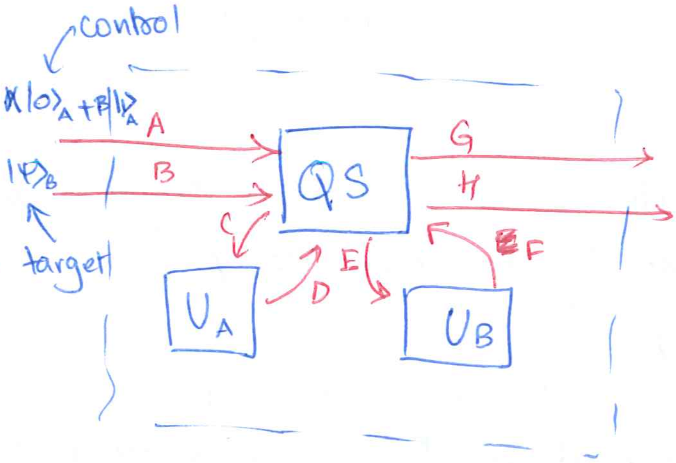
- Composition: parallel, sequence, loops ↖ from output at earlier time to input at later time.

$\hat{\Omega}$ ← also a CB.



→ CBs closed under composition (independent of order of composition)

- QS as a causal box



$$\left.\begin{aligned} &\alpha|0\rangle_G\, U_B U_A |\psi\rangle_H \\ &+ \beta|0\rangle_G\, U_A U_B |\psi\rangle_H \end{aligned}\right\}$$

Sequence of steps at times $t = 1, 2, 3, 4, 5$.

$t=1$: QS_1 stores control (wire A) in internal quantum memory M & sends target (wire B) to U_A or U_B depending on control & $|\Omega\rangle$ to the other unitary.

$$QS_1\, |0\rangle_A \otimes |\psi\rangle_B = |0\rangle_M \otimes |\Omega\rangle_C \otimes |\psi\rangle_E$$
$$QS_1\, |1\rangle_A \otimes |\psi\rangle_B = |1\rangle_M \otimes |\psi\rangle_C \otimes |\Omega\rangle_E$$

$t=2$: U's get applied

$t=3$: QS_3 forwards state from ~~...~~ $D \to E$ and $F \to C$

$$QS_3\, |\phi\rangle_D \otimes |\phi'\rangle_F = |\phi'\rangle_C \otimes |\phi\rangle_E$$ (note: one of ϕ or ϕ' will be the vacuum state Ω)

$t=4$: U's applied again

$t=5$: Depending on memory state, QS_5 outputs message from either U_A or U_B along with control.

$$QS_5\, |0\rangle_M \otimes |\psi\rangle_D \otimes |\Omega\rangle_F = |0\rangle_G \otimes |\psi\rangle_H$$
$$QS_5\, |1\rangle_M \otimes |\Omega\rangle_D \otimes |\psi\rangle_F = |1\rangle_G \otimes |\psi\rangle_H$$

* Apply these operations to input state on A, B, will get the output state on GH as in Figure.

* Each Unitary applied only once (action on $|\Omega\rangle$ not counted).

* Quantum memory preserves coherence: can do in superposition

- Applications of CBs (Vilasini et. al 2019 NJP)
  - ~~Rel.~~ Explicitly models quantum tasks in space-time. ⇒ application in relativistic quantum cryptography (quantum agents distributed in space-time)
  - composable security of a very general class of tasks, also those involving superpos. of orders.

VI. Comparison of QS in frameworks: insights.

PM	CB
1) No global order assumed ⇒ can violate causal inequalities ⇒ can model superpos. of space-time geometries	1) Global order defined by poset T ⇒ cannot violate causal inequalities ⇒ cannot model superpos. of space-time geometries.
2) U_A, U_B in separate labs, agents have no control over W	2) Everything in one lab, full control (as in actual experiment).
3) Does not consider the vacuum state being exchanged.	3) Explicitly considers vacuum state $\|\Omega\rangle$.

⇒ QS can be embedded in a fixed space-time.

No 'causal' influences here: just fixed operations U_A & U_B, but QS is often referred to as an "indefinite causal structure".

→ But it is a superposition of temporal orders of U_A & U_B rather than any causal order.

↳ this is much clearer from the CB approach & the experiment.

↳ Also depends on what we choose to call an 'event'.

We show: The CJ representation of the QS as a causal box gives the process matrix of QS when the vacuum states & time-stamps are ignored.

(ref: my masters thesis, unpublished)

↳ PM formalism ignores the time information, vacuum states!

They are physically very different!

★ But QS is still interesting: computational, communication advantage over fixed order of operations.

★ What would constitute a superpos. of causal orders?

Answer: Gravitational Quantum switch (?) (GQS)

Same PM for QS & GQS but CBs cannot model GQS!