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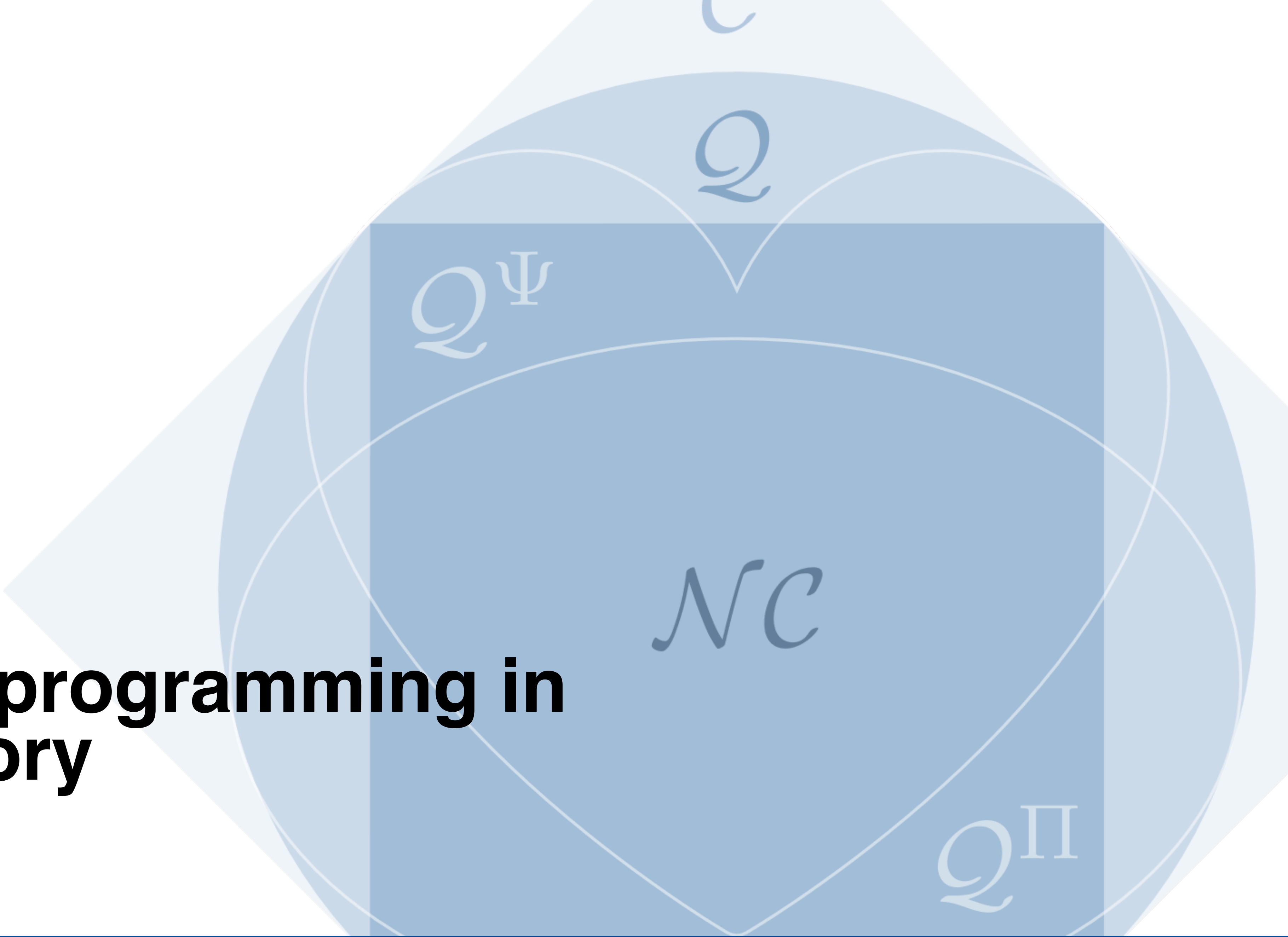

Semidefinite programming in quantum theory

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Solstice of Foundations 2022

Positive (semidefinite) operators are everywhere in quantum theory

$$A \geq 0 \iff v^T A v \geq 0 \forall v \in \mathscr{H}$$

$$\rho$$

Density operators

$$\{E_1, \ldots, E_n\}$$

POVMs

Semidefinite programming let us optimise linear functions over positive semidefinite matrices

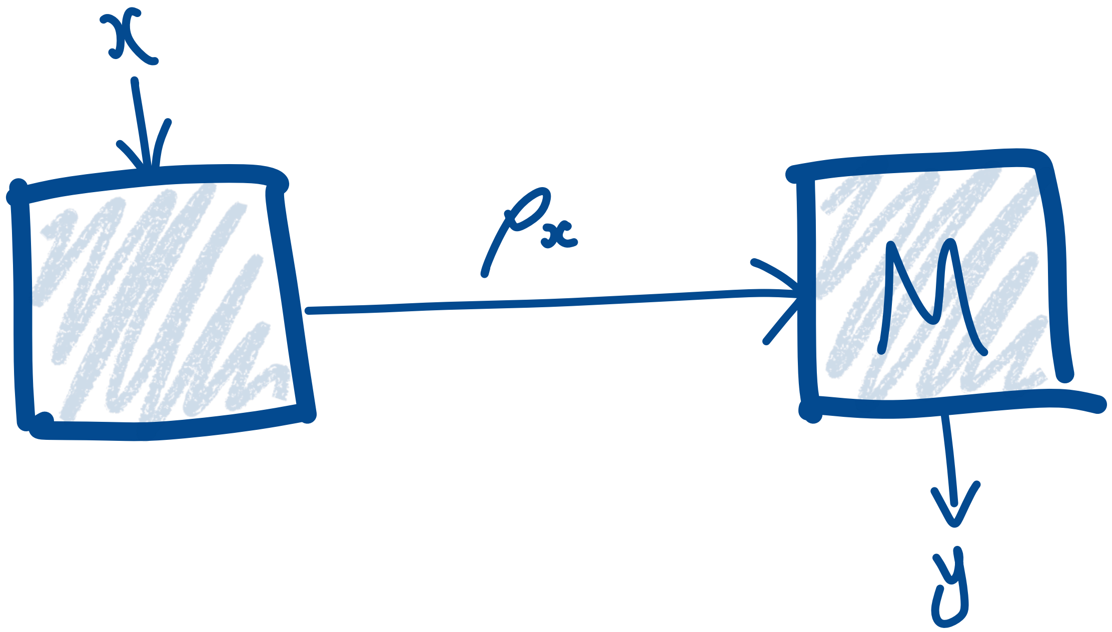
Classic reference [BV04], Quantum lecture notes [SV15]

State discrimination

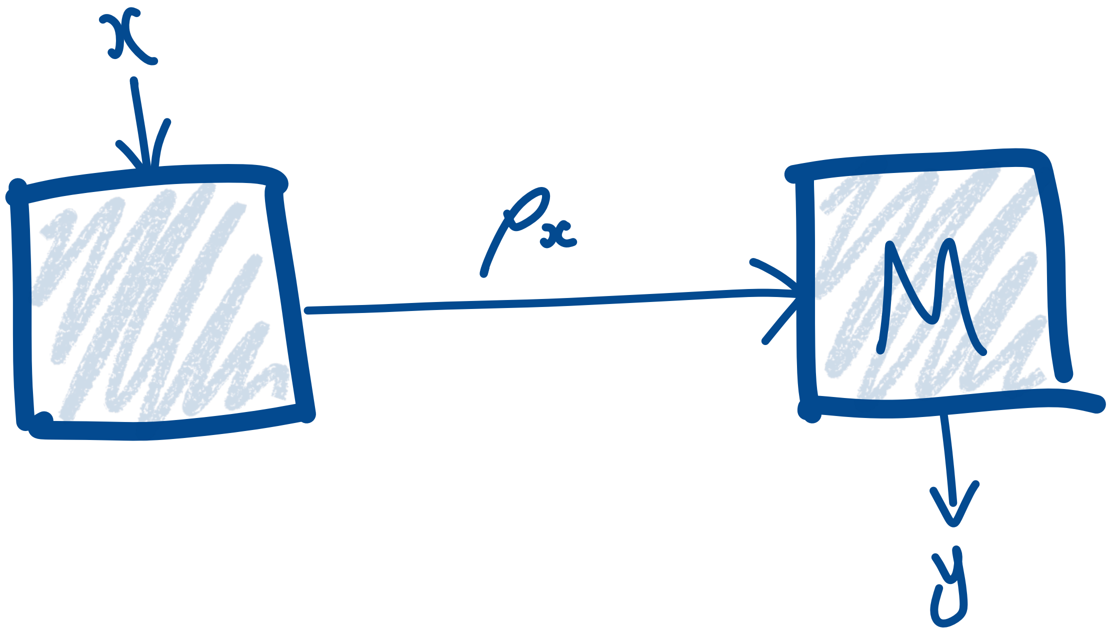
Task:

Discriminate $\rho_1, \rho_2, \ldots, \rho_n$

with a measurement $M = \{M_1, \ldots, M_n\}$



State discrimination



Probability of guessing correctly:

$$\frac{1}{n}\sum_{x=1}^{n}\mathrm{Tr}(M_x\rho_x)$$

Best chance

$$\max_{M_x\in H_d} \quad \sum_x \mathrm{Tr}(M_x\rho_x)$$
$$\text{s.t.} \quad \sum_x M_x = \mathbb{I}_d$$
$$M_x \geq 0$$

This is an SDP!

Notation:
H_d - $d\times d$ Hermitian matrices
S_d - $d\times d$ symmetric matrices

SEMIDEFINITE PROGRAM

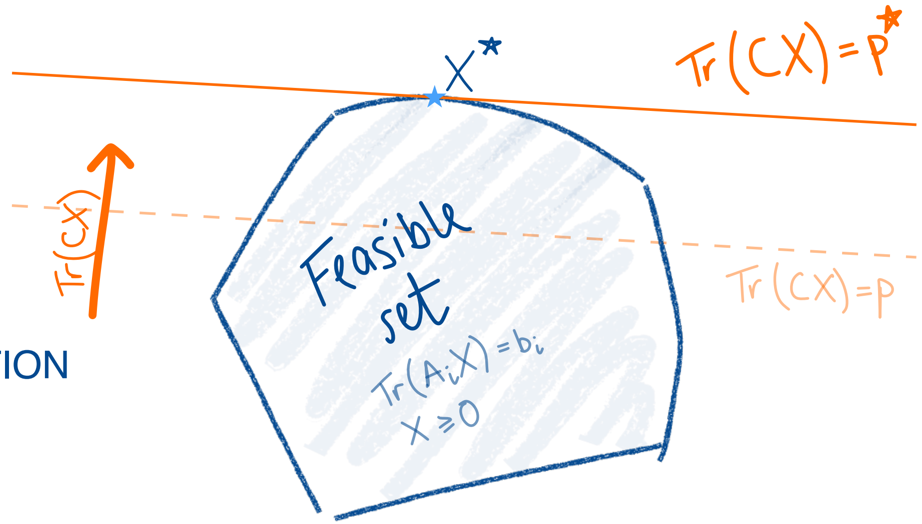
LINEAR PROGRAM

$$\max_{X \in H_d} \quad \mathrm{Tr}(CX)$$ LINEAR OBJECTIVE FUNCTION

$$\text{s.t.} \quad \mathrm{Tr}(A_i X) = b_i$$ LINEAR CONSTRAINTS

$$+ \quad X \geq 0$$ SEMIDEFINITE CONSTRAINT

$C, A_i \in H_d, b_i \in \mathbb{R}$

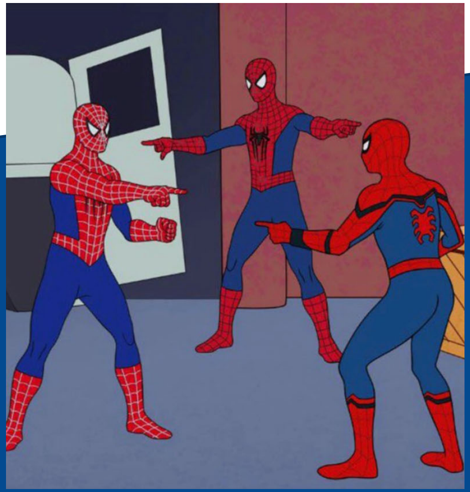


$$\max_{\mathbf{x} \in \mathbb{R}^d} \quad \mathbf{c} \cdot \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} = \mathbf{b}$$

$$x_1 F_1 + \cdots + x_d F_d + G \geq 0$$

$A \in \mathbb{R}^{n \times d}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^d, F_1, \ldots, F_d, G \in S_d$



$$\max_{X \in S_d} \quad \mathrm{Tr}(CX)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i X) = b_i$$

$$X \geq 0$$

$C, A_i \in S_d, b_i \in \mathbb{R}$

SEMIDEFINITE PROGRAM

LINEAR PROGRAM

$$\max_{X \in H_d} \quad \mathrm{Tr}(CX)$$

$$\text{s.t.} \quad \mathrm{Tr}(A_i X) = b_i$$

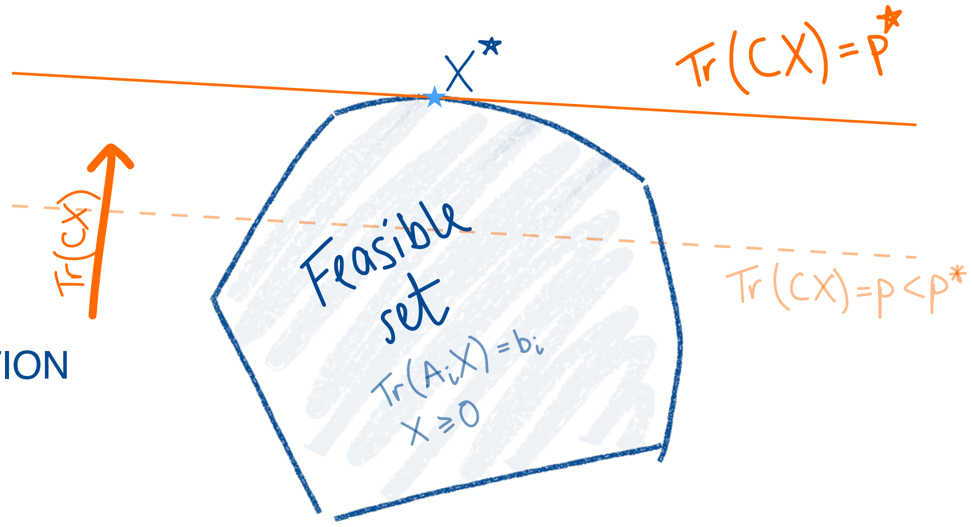
$+$

$$X \geq 0$$

LINEAR OBJECTIVE FUNCTION

LINEAR CONSTRAINTS

SEMIDEFINITE CONSTRAINT



$C, A_i \in H_d, b_i \in \mathbb{R}$

Ex: Put this SDP in canonical form

$$\max_{M_x \in H_d} \quad \sum_x \mathrm{Tr}(M_x \rho_x)$$

$$\text{s.t.} \quad \sum_x M_x = \mathbb{I}_d$$

$$M_x \geq 0$$

Why are we looking for SDPs?

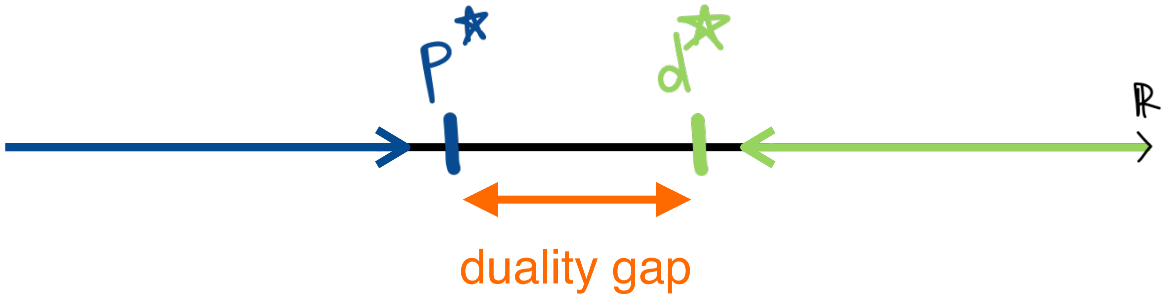
- Efficiently solvable
- Interior point methods
- At worst polynomial time in the size of X
- More efficient algorithms may exist

$$\max_{X \in H_d} \quad \mathrm{Tr}(CX)$$
$$\text{s.t.} \quad \mathrm{Tr}(A_i X) = b_i$$
$$X \geq 0$$

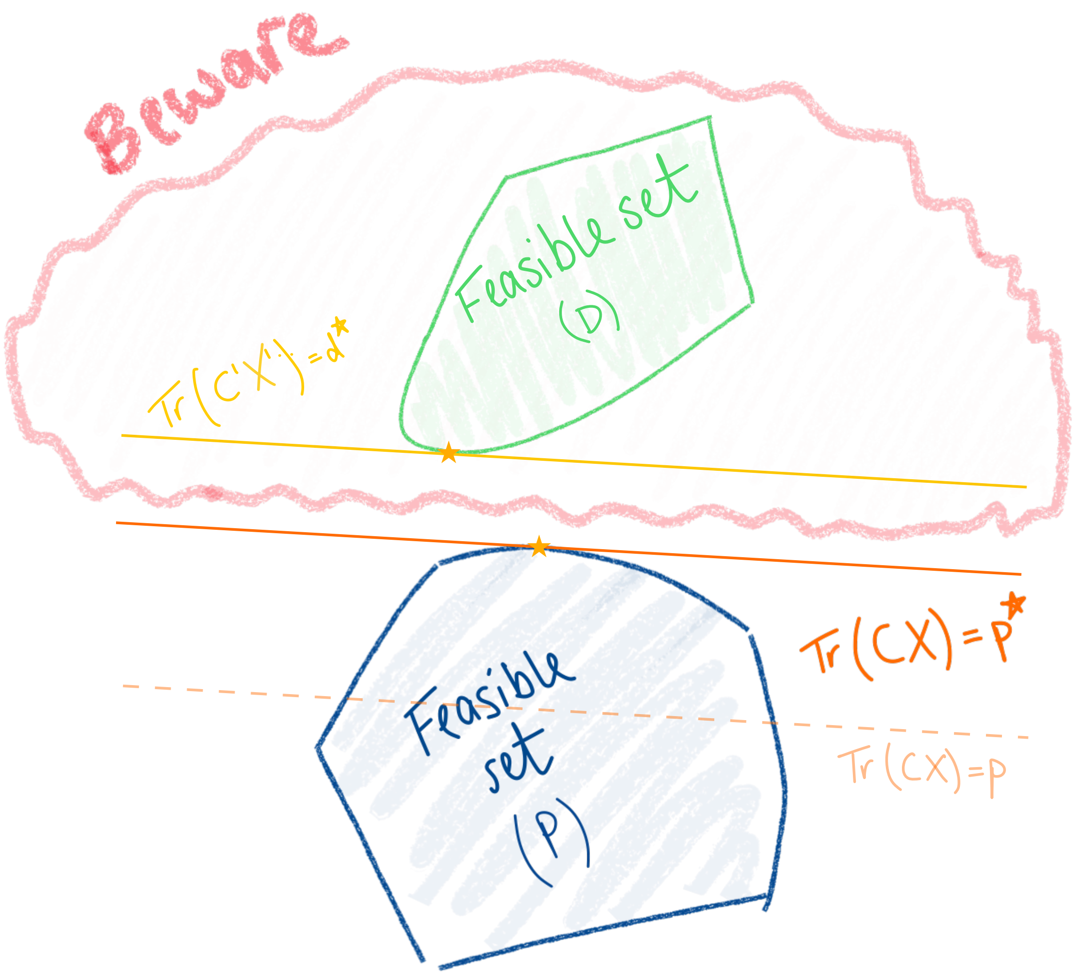
Important part of the theory of SDPs:

Dual SDPs

- Every SDP has a dual SDP
- Feasible point in *primal* SDP (P) → lower bound for $p^\star$
- Feasible point in *dual* SDP (D) → upper bound for $p^\star$
- The best upper bound $d^\star \geq p^\star$



- For some SDPs the duality gap is zero: $p^\star = d^\star$
- E.g. if *Slater's condition* is satisfied
- Then the dual provides a *certificate* of the optimal value of the primal



There exists a point in the relative interior of the feasible set, where the constraints hold strictly

Dual SDPs

$C, A_i \in S_d, b_i \in \mathbb{R}$

PRIMAL

$$\max_{X \in S_d} \quad \mathrm{Tr}(CX) = p^\star$$
$$\text{s.t.} \quad \mathrm{Tr}(A_i X) = b_i$$
$$X \geq 0$$

Expression to approximate $p^\star$
- from above
- without constraints
- new variables
- SDP friendly

Unconstrained

ADD

X not in feasible set:
punishment :(

X in feasible set:
reward :)

Lagrangian dual function

$$\sup_{X \in S_d} \quad \mathrm{Tr}(CX) + \mathrm{Tr}(YX) + \sum_i y_i \left[\mathrm{Tr}(A_i X) - b_i\right] \quad = g(Y, y_1, \ldots, y_n) \quad \geq p^\star$$

- If $X \not\geq 0$: negative for some $Y \geq 0$
- If $X \geq 0$: non-negative for any $Y \geq 0$

- If $\mathrm{Tr}(A_i X) \neq b$: negative for some $y_i \in \mathbb{R}$
- If $\mathrm{Tr}(A_i X) = b$: 0

To find the best (i.e. lowest) upper bound we minimise

Minimising $g(Y, y_1, \ldots, y_n)$

$$\min_{Y \in H_d, y_i \in \mathbb{R}} \quad \sup_{X \in H_d} \mathrm{Tr}(CX) + \mathrm{Tr}(YX) + \sum_i y_i \left[\mathrm{Tr}(A_i X) - b_i\right]$$

$$\text{s.t.} \quad Y \geq 0$$

$$\min_{Y \in H_d, y_i \in \mathbb{R}} \quad \sup_{X \in H_d} \mathrm{Tr}\left[\left(C + Y + \sum_i y_i A_i\right) X\right] - \sum_i b_i y_i$$

$$\text{s.t.} \quad Y \geq 0$$

unbounded unless $C + Y + \sum_i y_i A_i = 0$

$$\min_{Y \in H_d, y_i \in \mathbb{R}} \quad -\sum_i b_i y_i$$

$$\text{s.t.} \quad Y \geq 0, \quad C + Y + \sum_i y_i A_i = 0$$

$$C \leq -\sum_i y_i A_i$$

DUAL

$$\min_{y_i \in \mathbb{R}} \quad \sum_i b_i y_i$$

$$\text{s.t.} \quad \sum_i y_i A_i \geq C$$

State discrimination

PRIMAL

$$\max_{M_x \in H_d} \sum_x \mathrm{Tr}(M_x \rho_x)$$

$$\text{s.t.} \quad \sum_x M_x = \mathbb{1}_d$$

$$M_x \geq 0$$

DUAL

$$\min_{X \in H_d} \quad \mathrm{Tr}(X)$$

$$\text{s.t.} \quad X \geq \rho_x \quad \forall 1 \leq x \leq n$$

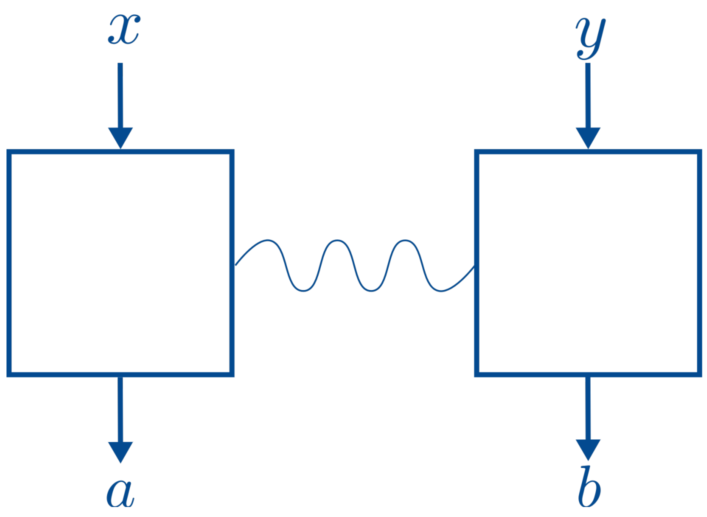
Exercise: derive the dual

$$\sup_{M_y \in H_d} \sum_y \mathrm{Tr}(M_y \rho_y) + \sum_y \mathrm{Tr}(Y_y M_y) + \mathrm{Tr}\left[X\left(\sum_y M_y - \mathbb{1}_d\right)\right]$$

Dual much simpler!

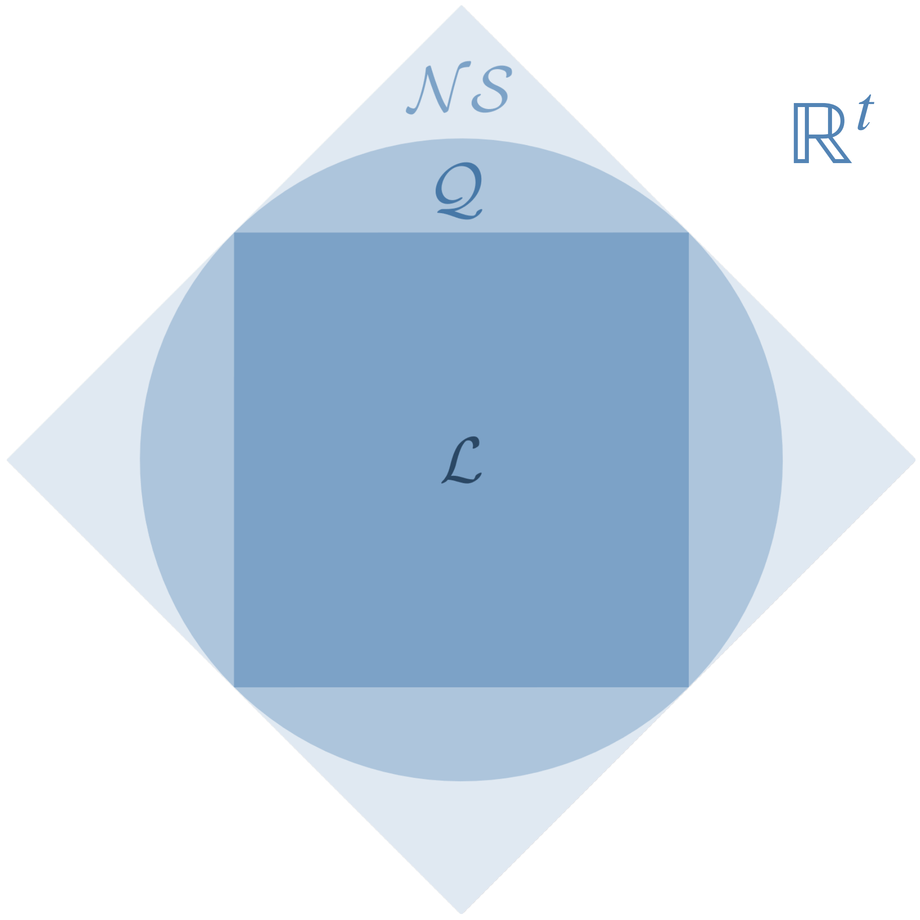
Bell nonlocality

Bell nonlocality (a recap)



Behaviour $$\mathbf{p}(ab\,|\,xy) = \begin{pmatrix} p(00\,|\,00) \\ p(01\,|\,00) \\ \vdots \end{pmatrix} \in \mathbb{R}^t$$

Local $$\mathbf{p} = \sum_j c_j \mathbf{d}_j$$



Is $\mathbf{p} \in \mathscr{L}$?

(P)

$$\max_{c_j} \quad 0$$

$$\text{s.t.} \quad \underline{p} = \sum_j c_j \underline{d}_j$$

(behaviour: $\underline{p}$; deterministic behaviours: $\underline{d}_j$)

$$\sum_j c_j = 1$$

$$0 \leq c_j$$

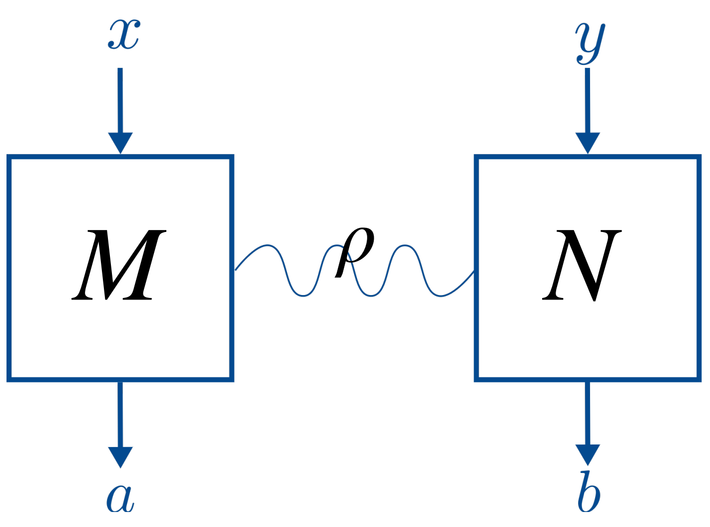
(D)

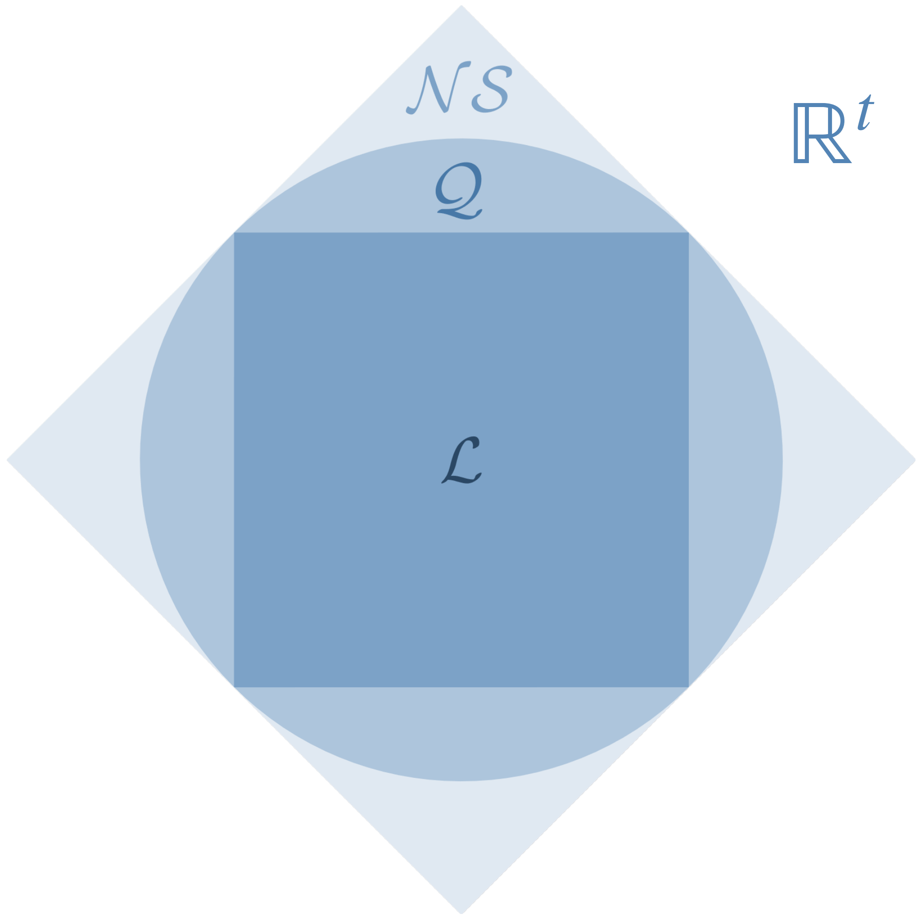
$$\max_{\underline{b},\, s} \quad \underline{b} \cdot \underline{p} - s$$

$$\text{s.t.} \quad \underline{b} \cdot \underline{d}_j \leq s$$

Input: nonlocal behaviour
Output: Bell equality violated by behaviour

Bell nonlocality (a recap)





Behaviour $\mathbf{p}(ab\,|\,xy) = \begin{pmatrix} p(00\,|\,00) \\ p(01\,|\,00) \\ \vdots \end{pmatrix} \in \mathbb{R}^t$

Quantum $p(ab\,|\,xy) = \mathrm{Tr}(M_a^x \otimes N_b^y \rho)$

Is p ∈ $\mathcal{Q}$?

dimension unbounded

$$\max_{M_a^x, N_b^y, \rho \in H_d, d \in \mathbb{N} \cup \infty} \quad 0$$

s.t.

non linear

$$\mathbf{p}(ab\,|\,xy) = \mathrm{Tr}(M_a^x \otimes N_b^y \rho)$$

semidef.

$$M_a^x, N_b^y, \rho \geq 0$$

$$\sum_a M_a^x = \sum_b N_b^y = \mathbb{1}_d$$

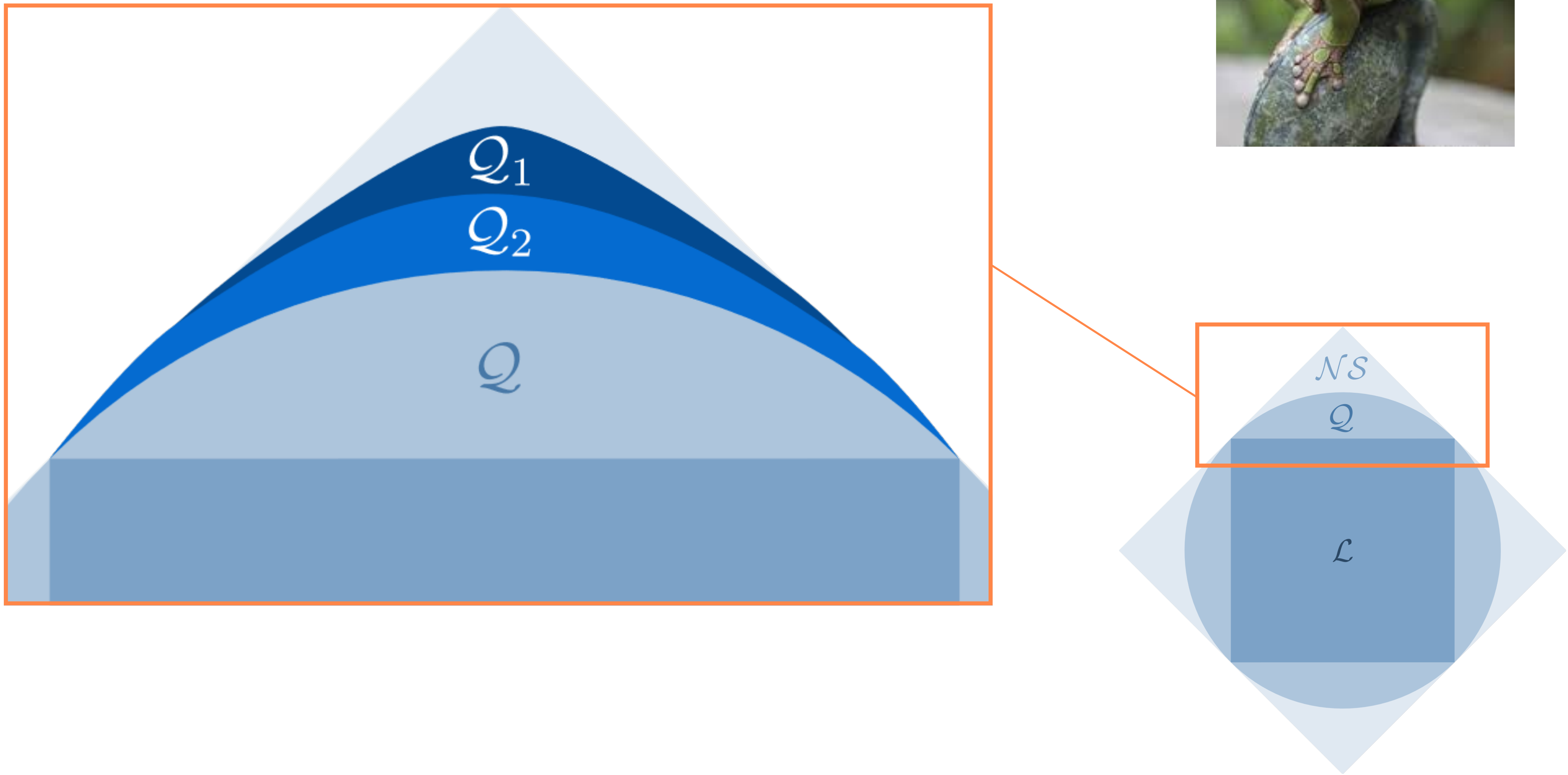
linear

$$\mathrm{Tr}(\rho) = 1$$

Problem is undecidable! [FMS21]

Nightmare! What do we do?

RELAX



1

If you can produce a quantum correlation with a mixed state and/or unsharp measurements

You can also produce it with a pure state and sharp measurements (on a possibly larger Hilbert space)

$$\mathbf{p} = \mathrm{Tr}(M_a^x \otimes N_b^y \rho) \iff \mathbf{p} = \langle \psi, P_a^x \otimes Q_b^y \psi \rangle$$

$$(P_a^x)^2 = P_a^x, \quad (Q_b^y)^2 = Q_b^y$$

2

Commuting quantum correlations, $\mathcal{Q}_c$

$$\mathbf{p} = \langle \psi, P_a^x \otimes Q_b^y \psi \rangle \implies \mathbf{p} = \langle \psi, P_a^x Q_b^y \psi \rangle$$

$$\not\Longleftarrow$$

$$[P_a^x, Q_b^y] = 0$$

Tsirelson's problem

[JNV20]

$\mathbf{p}(ab\,|\,xy) = \langle \psi, P_a^x Q_b^y \psi \rangle$

Projective mmts $\{P_a^x\}_a$ and $\{Q_b^y\}_b$ s.t. $[P_a^x, Q_b^y] = 0$, and pure state ψ

We can build a *Gram matrix* of the vectors $\left\{ \psi, P_a^x\psi, Q_b^y\psi \right\}$

$\{v_1, v_2, v_3, \dots\}$

$(\Gamma)_{i,j} = \langle v_i, v_j \rangle$

$$\Gamma = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \langle v_1, v_3\rangle & \cdots \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \cdots & \\ & & & \end{pmatrix} \geq 0$$

$\mathbf{p}(ab\,|\,xy) = \langle \psi, P_a^x Q_b^y \psi \rangle$ Projective mmts $\{P_a^x\}_a$ and $\{Q_b^y\}_b$ s.t. $[P_a^x, Q_b^y] = 0$, and pure state ψ

We can build a *Gram matrix* of the vectors $\left\{ \psi, P_a^x \psi, Q_b^y \psi \right\}$ $\langle P_a^x \psi, Q_b^y \psi \rangle = \langle \psi, P_a^x Q_b^y \psi \rangle$

	$\mathbb{1}$	P_0^0	P_1^0	Q_0^0	Q_1^0	P_0^1	P_1^1	…
$\mathbb{1}$	$\langle \mathbb{1} \rangle$	$\langle P_0^0 \rangle$	$\langle P_1^0 \rangle$	$\langle Q_0^0 \rangle$	$\langle Q_1^0 \rangle$	$\langle P_0^1 \rangle$	$\langle P_1^1 \rangle$	
P_0^0	$\langle P_0^0 \rangle$	$\langle P_0^0 \rangle$	$\langle P_0^0 P_1^0 \rangle$	$\langle P_0^0 Q_0^0 \rangle$	$\langle P_0^0 Q_1^0 \rangle$	$\langle P_0^0 P_0^1 \rangle$	$\langle P_0^0 P_1^1 \rangle$	
P_1^0	$\langle P_1^0 \rangle$	$\langle P_1^0 P_0^0 \rangle$	$\langle P_1^0 \rangle$	$\langle P_1^0 Q_0^0 \rangle$	$\langle P_1^0 Q_1^0 \rangle$	$\langle P_1^0 P_0^1 \rangle$	$\langle P_1^0 P_1^1 \rangle$	
Q_0^0	$\langle Q_0^0 \rangle$	$\langle Q_0^0 P_0^0 \rangle$	$\langle Q_0^0 P_1^0 \rangle$	$\langle Q_0^0 \rangle$	$\langle Q_0^0 Q_1^0 \rangle$	$\langle Q_0^0 P_0^1 \rangle$	$\langle Q_0^0 P_1^1 \rangle$	
Q_1^0	$\langle Q_1^0 \rangle$	$\langle Q_1^0 P_0^0 \rangle$	$\langle Q_1^0 P_1^0 \rangle$	$\langle Q_1^0 Q_0^0 \rangle$	$\langle Q_1^0 \rangle$	$\langle Q_1^0 P_0^1 \rangle$	$\langle Q_1^0 P_1^1 \rangle$	
⋮								

$\mathbf{p}(ab\,|\,xy) = \langle \psi, P_a^x Q_b^y \psi \rangle$ Projective mmts $\{P_a^x\}_a$ and $\{Q_b^y\}_b$ s.t. $[P_a^x, Q_b^y] = 0$, and pure state ψ

$\langle \psi, \psi \rangle = 1$

	$\mathbb{1}$	P_0^0	P_1^0	Q_0^0	Q_1^0	P_0^1	P_1^1	…
$\mathbb{1}$	$\langle \mathbb{1} \rangle$ $\mathbf{1}$	$\langle P_0^0 \rangle$	$\langle P_1^0 \rangle$	$\langle Q_0^0 \rangle$	$\langle Q_1^0 \rangle$	$\langle P_0^1 \rangle$	$\langle P_1^1 \rangle$	
P_0^0	$\langle P_0^0 \rangle$	$\langle P_0^0 \rangle$	$\langle P_0^0 P_1^0 \rangle$	$\langle P_0^0 Q_0^0 \rangle$	$\langle P_0^0 Q_1^0 \rangle$	$\langle P_0^0 P_0^1 \rangle$	$\langle P_0^0 P_1^1 \rangle$	
P_1^0	$\langle P_1^0 \rangle$	$\langle P_1^0 P_0^0 \rangle$	$\langle P_1^0 \rangle$	$\langle P_1^0 Q_0^0 \rangle$	$\langle P_1^0 Q_1^0 \rangle$	$\langle P_1^0 P_0^1 \rangle$	$\langle P_1^0 P_1^1 \rangle$	
Q_0^0	$\langle Q_0^0 \rangle$	$\langle Q_0^0 P_0^0 \rangle$	$\langle Q_0^0 P_1^0 \rangle$	$\langle Q_0^0 \rangle$	$\langle Q_0^0 Q_1^0 \rangle$	$\langle Q_0^0 P_0^1 \rangle$	$\langle Q_0^0 P_1^1 \rangle$	
Q_1^0	$\langle Q_1^0 \rangle$	$\langle Q_1^0 P_0^0 \rangle$	$\langle Q_1^0 P_1^0 \rangle$	$\langle Q_1^0 Q_0^0 \rangle$	$\langle Q_1^0 \rangle$	$\langle Q_1^0 P_0^1 \rangle$	$\langle Q_1^0 P_1^1 \rangle$	
⋮								

$$\mathbf{p}(ab\,|\,xy) = \langle \psi, P^x_a Q^y_b \psi \rangle$$

Projective mmts $\{P^x_a\}_a$ and $\{Q^y_b\}_b$ s.t. $[P^x_a, Q^y_b] = 0$, and pure state ψ

$$\langle P^0_0 \rangle = \langle \psi, P^0_0 \psi \rangle = \sum_b \langle \psi, P^0_0 Q^y_b \psi \rangle = \sum_b p(0b\,|\,0y) = p_A(0\,|\,0)$$

	$\mathbb{1}$	P^0_0	P^0_1	Q^0_0	Q^0_1	P^1_0	P^1_1	…
$\mathbb{1}$	1	$p_A(0\|0)$ $\langle P^0_0\rangle$	$p_A(1\|0)$ $\langle P^0_1\rangle$	$p_B(0\|0)$ $\langle Q^0_0\rangle$	$p_B(1\|0)$ $\langle Q^0_1\rangle$	$p_A(0\|1)$ $\langle P^1_0\rangle$	$p_A(1\|1)$ $\langle P^1_1\rangle$	
P^0_0	$p_A(0\|0)$ $\langle P^0_0\rangle$	$p_A(0\|0)$ $\langle P^0_0\rangle$	$\langle P^0_0 P^0_1\rangle$	$\langle P^0_0 Q^0_0\rangle$	$\langle P^0_0 Q^0_1\rangle$	$\langle P^0_0 P^1_0\rangle$	$\langle P^0_0 P^1_1\rangle$	
P^0_1	$p_A(1\|0)$ $\langle P^0_1\rangle$	$\langle P^0_1 P^0_0\rangle$	$p_A(1\|0)$ $\langle P^0_1\rangle$	$\langle P^0_1 Q^0_0\rangle$	$\langle P^0_1 Q^0_1\rangle$	$\langle P^0_1 P^1_0\rangle$	$\langle P^0_1 P^1_1\rangle$	
Q^0_0	$p_B(0\|0)$ $\langle Q^0_0\rangle$	$\langle Q^0_0 P^0_0\rangle$	$\langle Q^0_0 P^0_1\rangle$	$p_B(0\|0)$ $\langle Q^0_0\rangle$	$\langle Q^0_0 Q^0_1\rangle$	$\langle Q^0_0 P^1_0\rangle$	$\langle Q^0_0 P^1_1\rangle$	
Q^0_1	$p_B(1\|0)$ $\langle Q^0_1\rangle$	$\langle Q^0_1 P^0_0\rangle$	$\langle Q^0_1 P^0_1\rangle$	$\langle Q^0_1 Q^0_0\rangle$	$p_B(1\|0)$ $\langle Q^0_1\rangle$	$\langle Q^0_1 P^1_0\rangle$	$\langle Q^0_1 P^1_1\rangle$	
⋮								

$\mathbf{p}(ab|xy) = \langle \psi, P^x_a Q^y_b \psi \rangle$ Projective mmts $\{P^x_a\}_a$ and $\{Q^y_b\}_b$ s.t. $[P^x_a, Q^y_b] = 0$, and pure state ψ

$\langle P^0_0 Q^0_0 \rangle = \langle \psi, P^0_0 Q^0_0 \psi \rangle = p(00|00)$

	$\mathbb{1}$	P^0_0	P^0_1	Q^0_0	Q^0_1	P^1_0	P^1_1	…
$\mathbb{1}$	1	$p_A(0\|0)$	$p_A(1\|0)$	$p_B(0\|0)$	$p_B(1\|0)$	$p_A(0\|1)$	$p_A(1\|1)$	
P^0_0	$p_A(0\|0)$	$p_A(0\|0)$	$\langle P^0_0 P^0_1 \rangle$	$p(00\|00)$ $\langle P^0_0 Q^0_0 \rangle$	$p(01\|00)$ $\langle P^0_0 Q^0_1 \rangle$	$\langle P^0_0 P^1_0 \rangle$	$\langle P^0_0 P^1_1 \rangle$	
P^0_1	$p_A(1\|0)$	$\langle P^0_1 P^0_0 \rangle$	$p_A(1\|0)$	$p(10\|00)$ $\langle P^0_1 Q^0_0 \rangle$	$p(11\|00)$ $\langle P^0_1 Q^0_1 \rangle$	$\langle P^0_1 P^1_0 \rangle$	$\langle P^0_1 P^1_1 \rangle$	
Q^0_0	$p_B(0\|0)$	$\langle Q^0_0 P^0_0 \rangle$	$\langle Q^0_0 P^0_1 \rangle$	$p_B(0\|0)$	$\langle Q^0_0 Q^0_1 \rangle$	$p(00\|10)$ $\langle Q^0_0 P^1_0 \rangle$	$p(10\|10)$ $\langle Q^0_0 P^1_1 \rangle$	
Q^0_1	$p_B(1\|0)$	$\langle Q^0_1 P^0_0 \rangle$	$\langle Q^0_1 P^0_1 \rangle$	$\langle Q^0_1 Q^0_0 \rangle$	$p_B(1\|0)$	$p(01\|10)$ $\langle Q^0_1 P^1_0 \rangle$	$p(11\|10)$ $\langle Q^0_1 P^1_1 \rangle$	
⋮								

$\mathbf{p}(ab\,|\,xy) = \langle \psi, P^x_a Q^y_b \psi \rangle$ Projective mmts $\{P^x_a\}_a$ and $\{Q^y_b\}_b$ s.t. $[P^x_a, Q^y_b] = 0$, and pure state ψ

	▯	P^0_0	P^0_1	Q^0_0	Q^0_1	P^1_0	P^1_1	…
▯	1	$p_A(0\|0)$	$p_A(1\|0)$	$p_B(0\|0)$	$p_B(1\|0)$	$p_A(0\|1)$	$p_A(1\|1)$	
P^0_0	$p_A(0\|0)$	$p_A(0\|0)$	$\langle P^0_0 P^0_1 \rangle$ 0	$p(00\|00)$	$p(01\|00)$	$\langle P^0_0 P^1_0 \rangle$	$\langle P^0_0 P^1_1 \rangle$	
P^0_1	$p_A(1\|0)$	$\langle P^0_1 P^0_0 \rangle$	$p_A(1\|0)$	$p(10\|00)$	$p(11\|00)$	$\langle P^0_1 P^1_0 \rangle$	$\langle P^0_1 P^1_1 \rangle$	
Q^0_0	$p_B(0\|0)$	$\langle Q^0_0 P^0_0 \rangle$	$\langle Q^0_0 P^0_1 \rangle$	$p_B(0\|0)$	$\langle Q^0_0 Q^0_1 \rangle$ 0	$p(00\|10)$	$p(10\|10)$	
Q^0_1	$p_B(1\|0)$	$\langle Q^0_1 P^0_0 \rangle$	$\langle Q^0_1 P^0_1 \rangle$	$\langle Q^0_1 Q^0_0 \rangle$	$p_B(1\|0)$	$p(01\|10)$	$p(11\|10)$	
⋮								

$\mathbf{p}(ab\,|\,xy) = \langle \psi, P^x_a Q^y_b \psi \rangle$ Projective mmts $\{P^x_a\}_a$ and $\{Q^y_b\}_b$ s.t. $[P^x_a, Q^y_b] = 0$, and pure state ψ

Moment matrix

$$\Gamma$$

	$\mathbb{1}$	P^0_0	P^0_1	Q^0_0	Q^0_1	P^1_0	P^1_1	…
$\mathbb{1}$	1	$p_A(0\|0)$	$p_A(1\|0)$	$p_B(0\|0)$	$p_B(1\|0)$	$p_A(0\|1)$	$p_A(1\|1)$	
P^0_0	$p_A(0\|0)$	$p_A(0\|0)$	0	$p(00\|00)$	$p(01\|00)$	$\langle P^0_0 P^1_0 \rangle$	$\langle P^0_0 P^1_1 \rangle$	
P^0_1	$p_A(1\|0)$	$\langle P^0_1 P^0_0 \rangle$ / 0	$p_A(1\|0)$	$p(10\|00)$	$p(11\|00)$	$\langle P^0_1 P^1_0 \rangle$	$\langle P^0_1 P^1_1 \rangle$	
Q^0_0	$p_B(0\|0)$	$p(00\|00)$ / $\langle Q^0_0 P^0_0 \rangle$	$p(10\|00)$ / $\langle Q^0_0 P^0_1 \rangle$	$p_B(0\|0)$	0	$p(00\|10)$	$p(10\|10)$	
Q^0_1	$p_B(1\|0)$	$p(01\|00)$ / $\langle Q^0_1 P^0_0 \rangle$	$p(11\|00)$ / $\langle Q^0_1 P^0_1 \rangle$	0 / $\langle Q^0_1 Q^0_0 \rangle$	$p_B(1\|0)$	$p(01\|10)$	$p(11\|10)$	
⋮								

$$\mathbf{p}(ab\,|\,xy) = \langle \psi, P_a^x Q_b^y \psi \rangle \quad [P_a^x, Q_b^y] = 0 \quad (P_a^x)^2 = P_a^x \quad (Q_b^y)^2 = Q_b^y \quad \sum_a P_a^x = \sum_b Q_b^y = \mathbb{1}$$

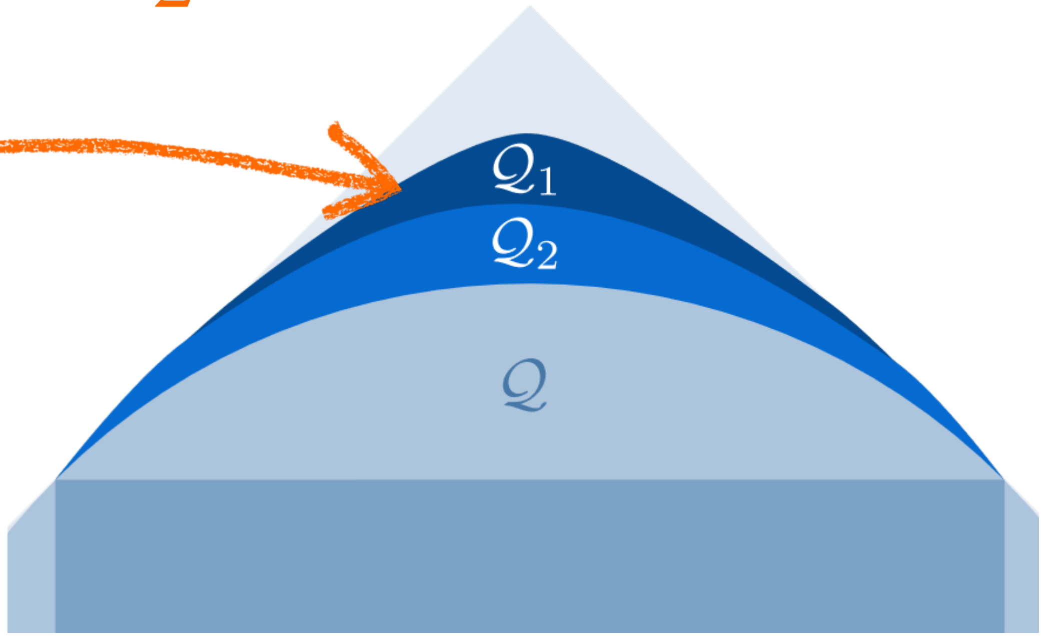
$\implies$ There exists Hermitian $\Gamma \geq 0$ such that $(\Gamma)_{i,j} = \mathbf{c}_{i,j} \cdot \mathbf{p}(ab\,|\,xy)$

$\iff$ There exists symmetric $\Gamma' \geq 0$ such that $(\Gamma')_{i,j} = \mathbf{c}_{i,j} \cdot \mathbf{p}(ab\,|\,xy)$

$$\Gamma' = \frac{\Gamma + \Gamma^*}{2}$$

Related to [MMG09]

$$\begin{aligned} \max_{\Gamma \in S_d} \quad & 0 \\ \text{s.t.} \quad & (\Gamma)_{i,j} = \mathbf{c}_{i,j} \cdot \mathbf{p}(ab\,|\,xy) \\ & \Gamma \geq 0 \end{aligned}$$



Also, maximise Bell functionals!

Bell functionals and operators

Bell functional: linear functional on $\mathbf{p} \in \mathbb{R}^t$

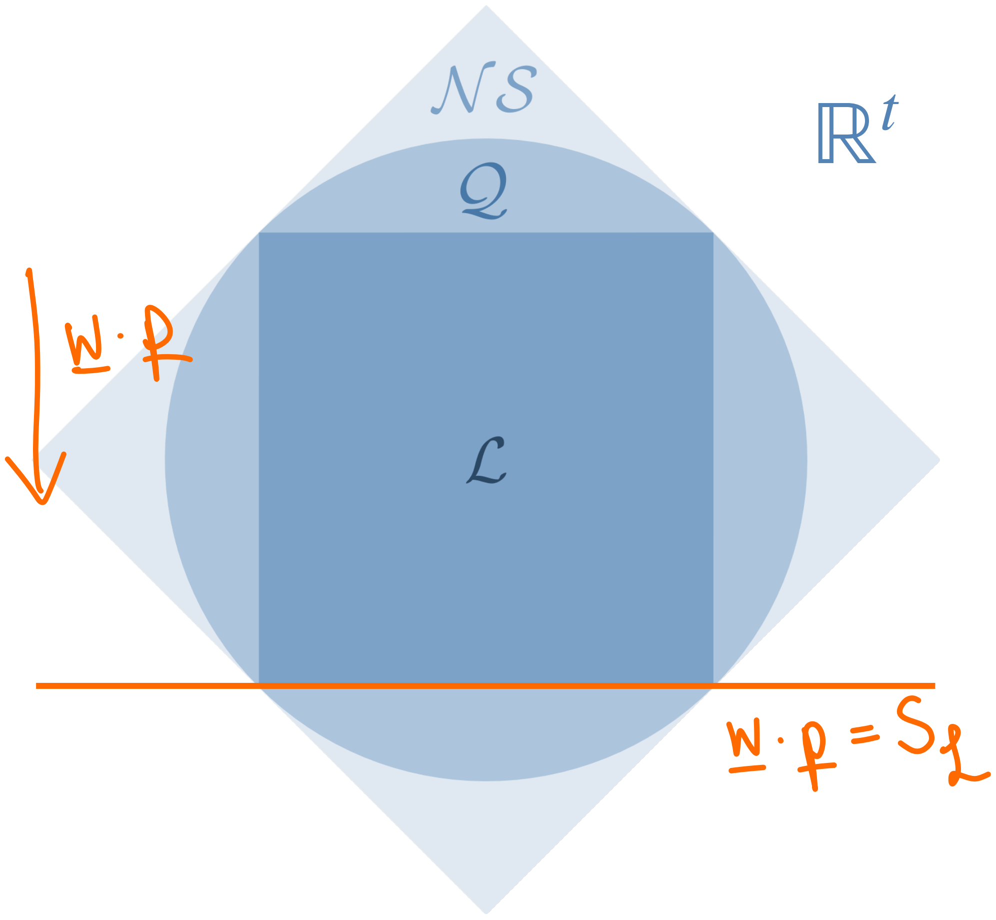
$$\mathbf{w} \cdot \mathbf{p} \in \mathbb{R}$$

Bell inequality: local bound on value of Bell functional

$$\mathbf{w} \cdot \mathbf{p} \leq S_{\mathscr{L}} \qquad \forall \mathbf{p} \in \mathscr{L}$$

Quantum bound of Bell functional

$$\mathbf{w} \cdot \mathbf{p} \leq S_{\mathscr{Q}}$$



Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	$\mathbb{1}$	P_0^0	P_1^0	P_0^1	P_1^1	Q_0^0	Q_1^0	Q_0^1	Q_1^1
$\mathbb{1}$									
P_0^0									
P_1^0									
P_0^1									
P_1^1									
Q_0^0									
Q_1^0									
Q_0^1									

Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	$\mathbb{1}$	P_0^0	P_1^0	P_0^1	P_1^1	Q_0^0	Q_1^0	Q_0^1	Q_1^1
$\mathbb{1}$	1	$\langle P_0^0 \rangle$	$\langle P_1^0 \rangle$	$= 1 - \langle P_0^0 \rangle$					
P_0^0	$\langle P_0^0 \rangle$	$\langle P_0^0 \rangle$	$\langle P_0^0 P_1^0 \rangle$	$= 0$					
P_1^0		$\vdots$							
P_0^1									
P_1^1									
Q_0^0	$\langle Q_0^0 \rangle$	$\langle Q_0^0 P_0^0 \rangle$	$\langle Q_0^0 P_1^0 \rangle$	$= \langle Q_0^0 \rangle - \langle Q_0^0 P_0^0 \rangle$					
Q_1^0		$\vdots$							
Q_0^1									

$$P_1^0 = \mathbb{1} - P_0^0$$

Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	$\mathbb{1}$	P_0^0	~~P_1^0~~	P_0^1	~~P_1^1~~	Q_0^0	~~Q_1^0~~	Q_0^1	~~Q_1^1~~
$\mathbb{1}$									
P_0^0									
~~P_1^0~~									
P_0^1									
~~P_1^1~~									
Q_0^0									
~~Q_1^0~~									
Q_0^1									

Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	$\mathbb{1}$	P_0^0	P_0^1	Q_0^0	Q_0^1
$\mathbb{1}$					
P_0^0					
P_0^1					
Q_0^0					
Q_0^1					

One operator for each measurement

Use observables instead

$$A_x = P_0^x - P_1^x$$

$$B_y = Q_0^y - Q_1^y$$

Neat CHSH inequality: $$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2$$

Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	▯	P_0^0	P_0^1	Q_0^0	Q_0^1
▯					
P_0^0					
P_0^1					
Q_0^0					
Q_0^1					

	▯	A_0	A_1	B_0	B_1
▯					
A_0					
A_1					
B_0					
B_1					

Example: CHSH scenario 2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	▯	A_0	A_1	B_0	B_1
▯	$\langle ▯ \rangle$	$\langle A_0 \rangle$	$\langle A_1 \rangle$	$\langle B_0 \rangle$	$\langle B_1 \rangle$
A_0	$\langle A_0 \rangle$	$\langle (A_0)^2 \rangle$	$\langle A_0 A_1 \rangle$	$\langle A_0 B_0 \rangle$	$\langle A_0 B_1 \rangle$
A_1	$\langle A_1 \rangle$	$\langle A_1 A_0 \rangle$	$\langle (A_1)^2 \rangle$	$\langle A_1 B_0 \rangle$	$\langle A_1 B_1 \rangle$
B_0	$\langle B_0 \rangle$	$\langle B_0 A_0 \rangle$	$\langle B_0 A_1 \rangle$	$\langle (B_0)^2 \rangle$	$\langle B_0 B_1 \rangle$
B_1	$\langle B_1 \rangle$	$\langle B_1 A_0 \rangle$	$\langle B_1 A_1 \rangle$	$\langle B_1 B_0 \rangle$	$\langle (B_1)^2 \rangle$

Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	$\mathbb{1}$	A_0	A_1	B_0	B_1
$\mathbb{1}$	1	$\langle A_0 \rangle$	$\langle A_1 \rangle$	$\langle B_0 \rangle$	$\langle B_1 \rangle$
A_0	$\langle A_0 \rangle$	1	$\langle A_0 A_1 \rangle$	$\langle A_0 B_0 \rangle$	$\langle A_0 B_1 \rangle$
A_1	$\langle A_1 \rangle$	$\langle A_1 A_0 \rangle$	1	$\langle A_1 B_0 \rangle$	$\langle A_1 B_1 \rangle$
B_0	$\langle B_0 \rangle$	$\langle B_0 A_0 \rangle$	$\langle B_0 A_1 \rangle$	1	$\langle B_0 B_1 \rangle$
B_1	$\langle B_1 \rangle$	$\langle B_1 A_0 \rangle$	$\langle B_1 A_1 \rangle$	$\langle B_1 B_0 \rangle$	1

Example: CHSH scenario

2 inputs 2 outputs $x, y, a, b \in \{0,1\}$

	$\mathbb{1}$	A_0	A_1	B_0	B_1
$\mathbb{1}$	1	$\langle A_0 \rangle$	$\langle A_1 \rangle$	$\langle B_0 \rangle$	$\langle B_1 \rangle$
A_0	$\langle A_0 \rangle$	1	$\langle A_0 A_1 \rangle$	$\langle A_0 B_0 \rangle$	$\langle A_0 B_1 \rangle$
A_1	$\langle A_1 \rangle$	$\langle A_1 A_0 \rangle$	1	$\langle A_1 B_0 \rangle$	$\langle A_1 B_1 \rangle$
B_0	$\langle B_0 \rangle$	$\langle B_0 A_0 \rangle$	$\langle B_0 A_1 \rangle$	1	$\langle B_0 B_1 \rangle$
B_1	$\langle B_1 \rangle$	$\langle B_1 A_0 \rangle$	$\langle B_1 A_1 \rangle$	$\langle B_1 B_0 \rangle$	1

$$\max_{\Gamma \in S_5} \quad \Gamma_{2,4} + \Gamma_{2,5} + \Gamma_{3,4} - \Gamma_{3,5}$$

$$\text{s.t.} \quad (\Gamma)_{j,j} = 1$$

$$\Gamma \geq 0$$

Neat CHSH inequality: $\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2$

twosquiteffect.m | upperbounds2.m | squitvert.m | minval.m | twosquitextremalstates.m | twosquitstate.m | CHSHviolation.m | actualset.m | oddpolverts.m | minaqbellfunctional.m | SOS.m | CHSH.m

```matlab
function maxval = CHSH
yalmip('clear');
gamma = sdpvar(5); %Initialising moment matrix as a 5x5 symmetric matrix
C = 0.5*[0,0,0,0,0;0,0,0,1,1;0,0,0,1,-1;0,1,1,0,0;0,1,-1,0,0]; %CHSH bell functional: Tr(C*Gamma) is the Bell operator
Con = [gamma>=0;gamma(1,1) == 1;gamma(2,2) == 1;gamma(3,3) == 1;gamma(4,4) == 1;gamma(5,5) == 1;]; % Constraints of the SDP, moment matrix must be positive and one on diagonals
eta = trace(C*gamma); %objective function
diagnostics = optimize([Con;], -eta , sdpsettings('solver', 'mosek','verbose',1)); %optimisation: -eta is the objective function since YALMIP minimises and we want to maximise
moment = value(gamma) %prints the optimal moment matrix
maxval = value(eta);% returns the maximum value of the CHSH functional in Q1
end
```

Matlab+YALMIP+MOSEK

Command Window

```
Interior-point solution summary
  Problem status  : PRIMAL_AND_DUAL_FEASIBLE
  Solution status : OPTIMAL
  Primal.  obj: 2.8284271247e+00    nrm: 1e+00    Viol.  con: 2e-13    var: 0e+00    barvar: 0e+00
  Dual.    obj: 2.8284271247e+00    nrm: 1e+00    Viol.  con: 0e+00    var: 3e-13    barvar: 2e-13
Optimizer summary
  Optimizer                 -                   time: 0.14
    Interior-point          - iterations : 4    time: 0.13
      Basis identification  -                   time: 0.00
        Primal              - iterations : 0    time: 0.00
        Dual                - iterations : 0    time: 0.00
        Clean primal        - iterations : 0    time: 0.00
        Clean dual          - iterations : 0    time: 0.00
    Simplex                 -                   time: 0.00
      Primal simplex        - iterations : 0    time: 0.00
      Dual simplex          - iterations : 0    time: 0.00
    Mixed integer           - relaxations: 0    time: 0.00


moment =

    1.0000         0         0         0         0
         0    1.0000   -0.0000    0.7071    0.7071
         0   -0.0000    1.0000    0.7071   -0.7071
         0    0.7071    0.7071    1.0000    0.0000
         0    0.7071   -0.7071    0.0000    1.0000


ans =

    2.8284
```

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 1 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \end{pmatrix}$$

$2\sqrt{2}$

Ex.: Find the maximum value in $\mathcal{Q}_1$ of the titled CHSH inequality for some $0 \leq \alpha \leq 2$

$$\alpha\langle A_0\rangle + \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle$$

$\mathbf{p}(ab\,|\,xy) = \langle \psi, P_a^x Q_b^y \psi \rangle$ Projective mmts $\{P_a^x\}_a$ and $\{Q_b^y\}_b$ s.t. $[P_a^x, Q_b^y] = 0$, and pure state ψ

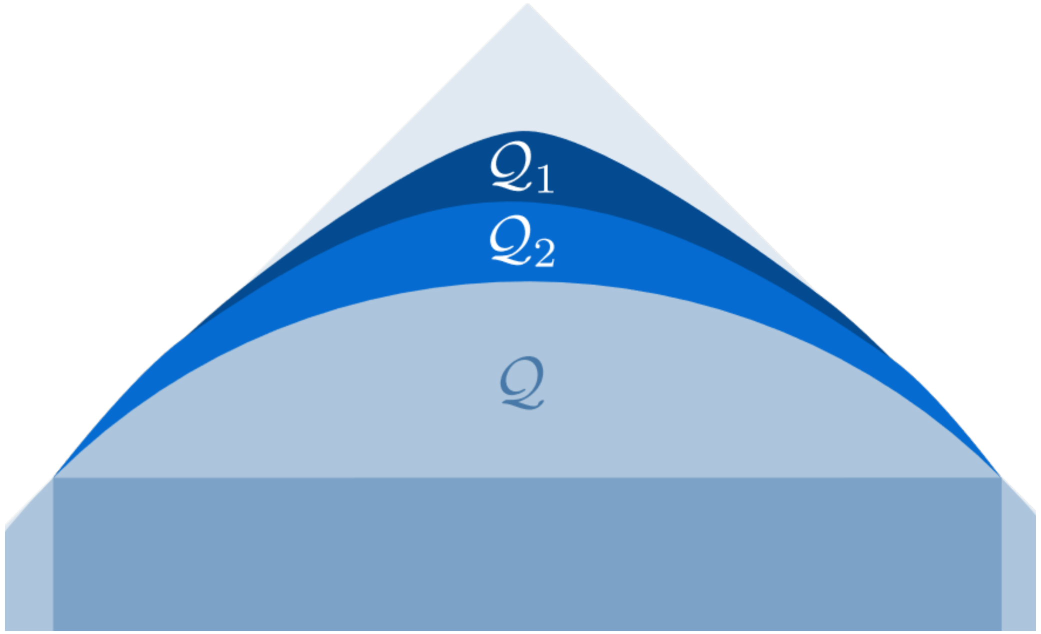
LEVEL 1

Gram matrix of the vectors $\left\{ A\psi \mid A \in \{\mathbb{I}, P_a^x, Q_b^y\} \right\}$

LEVEL 2

Gram matrix of the vectors $\left\{ AB\psi \mid A, B \in \{\mathbb{I}, P_a^x, Q_b^y\} \right\}$

⋮

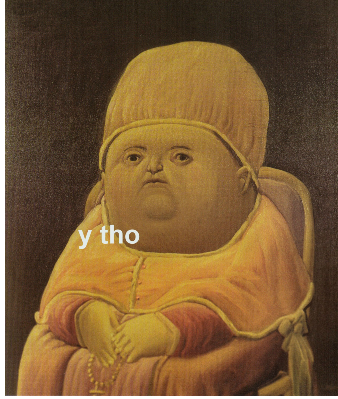


NPA hierarchy [NPA08] etc.

Converges to $\mathcal{Q}_C$

Implementation [Wittek15]

Where did this moment matrix come from? [NPA12]



Polynomial optimisation a la Lasserre

$$\min_{x,y\in\mathbb{R}} \; xy \quad \longrightarrow \quad m_{12}$$

Polynomials linear in new variables

$$\text{s.t.} \quad x^2 - x = 0$$

$$-y^2 + y + \frac{1}{2} \geq 0$$

Relationships between new variables e.g. $m_2 m_2 = m_{22}$ relaxed via positivity of a matrix

$$\begin{pmatrix} 1 & m_1 & m_2 \\ m_1 & m_1 & m_{12} \\ m_2 & m_{12} & m_{22} \end{pmatrix} \geq 0$$

$$\mathscr{B} = \{1, x, y, y^2, xy\} = \{m_0, m_1, m_2, m_{22}, m_{12}\}$$

Bell operators are noncommutative polynomials!

$$\langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle = \quad \langle \psi, (A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1)\psi\rangle$$

noncommutative polynomial

Cool things about NPA

- General approach for dealing with unbounded dimension
- Dual gives Sum-of-Squares (SOS) decomposition of "Bell operator" [IR21]

$$\langle\psi, (A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1)\psi\rangle$$

$$\langle\psi, W\psi\rangle \leq \lambda$$

largest eigenvalue of W

$$\langle\psi, W\psi\rangle \leq \langle\psi, \lambda\mathbb{I}\,\psi\rangle$$

λ is an upper bound on max quantum value of W $\Leftarrow$ $\lambda\mathbb{I} - W \geq 0$ holds ∀ λ ≥ largest eigenvalue

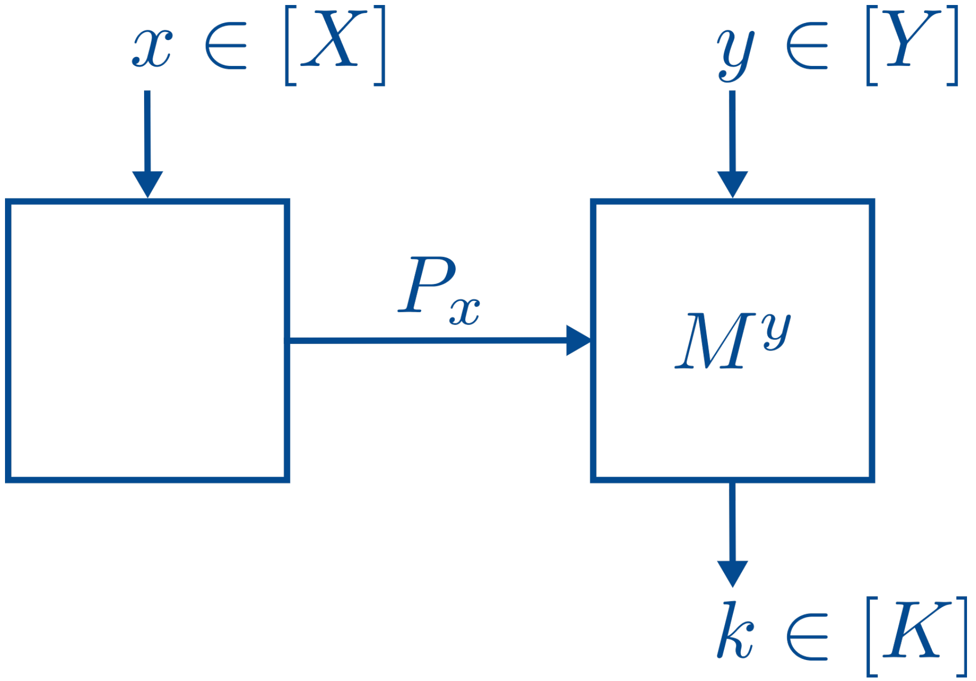
How to show $\lambda\mathbb{I} - W \geq 0$?

SOS-decomposition $\lambda\mathbb{I} - W = \sum_i P_i^+ P_i$

- Level 1+AB "Almost quantum correlations" [NGHA15]
- Solve the MUB problem? [NPA12]

Contextuality

Contextuality

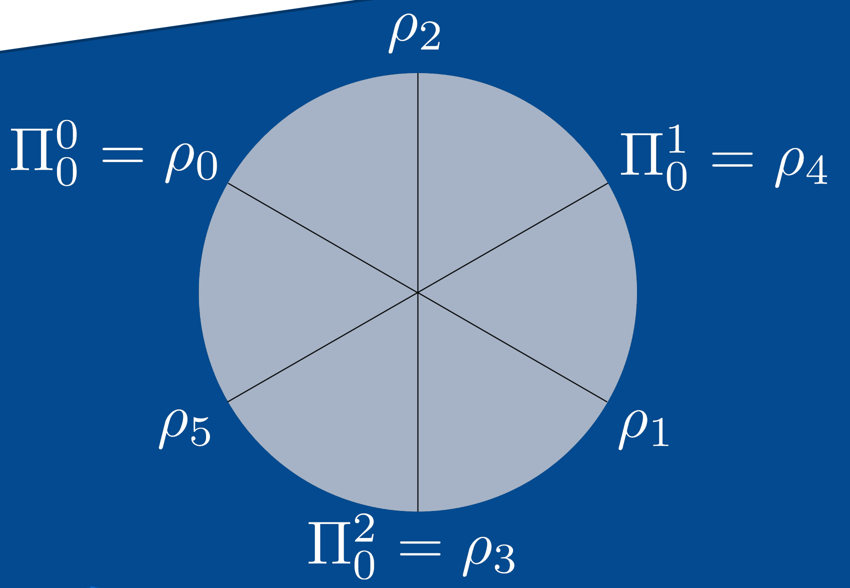


Contextuality scenario

$$T = (X, Y, K, \mathcal{OE}_P, \mathcal{OE}_M)$$

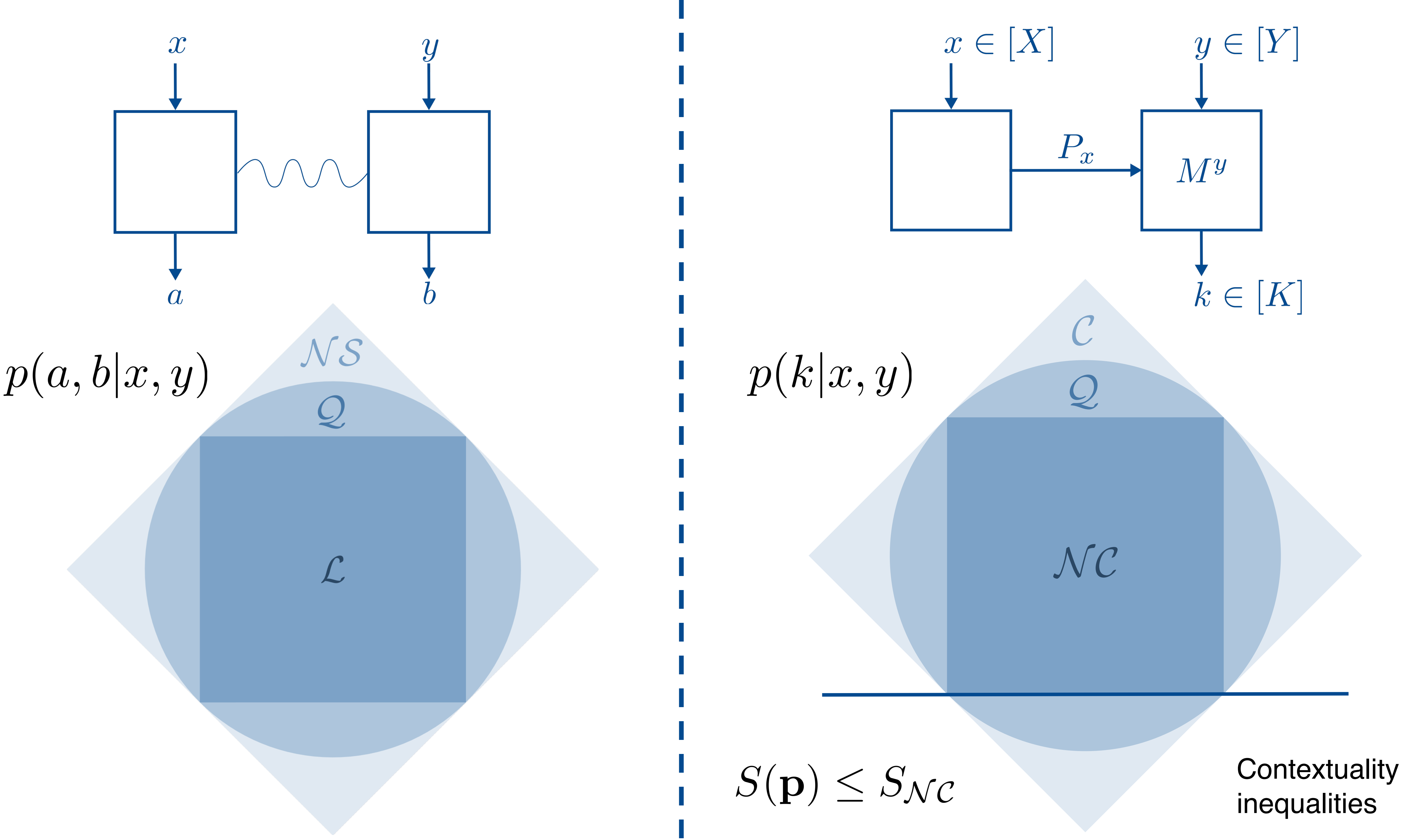
$$p(k|x,y) \qquad \mathbf{p} \in \mathbb{R}^{XYK}$$

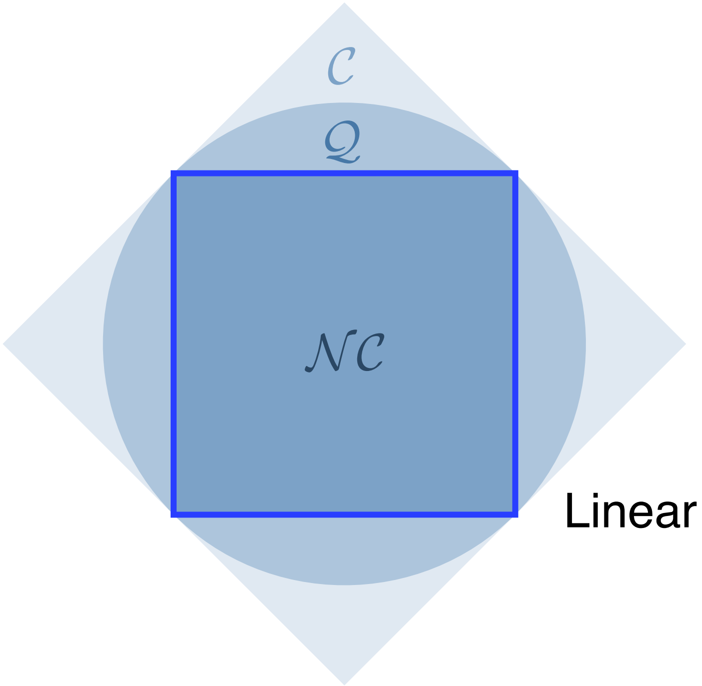
Example



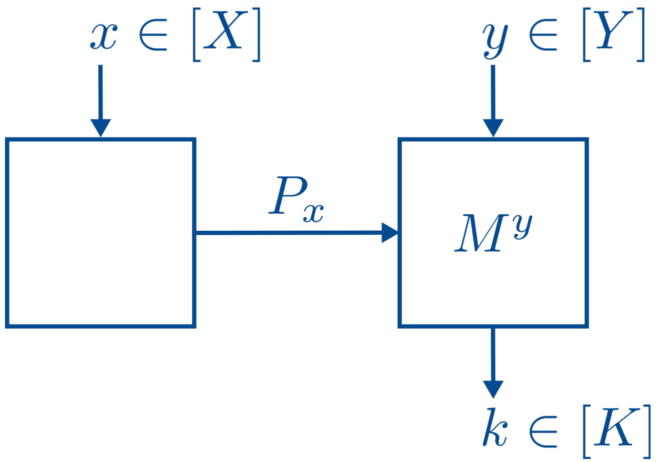
$$\tfrac{1}{2}(P_0 + P_1) \simeq \tfrac{1}{2}(P_2 + P_3) \simeq \tfrac{1}{2}(P_4 + P_5)$$

$$\tfrac{1}{3}([0|M_0] + [0|M_1] + [0|M_2]) \simeq \tfrac{1}{3}([1|M_0] + [1|M_1] + [1|M_2])$$


x
y
a
b
$p(a,b|x,y)$
$\mathcal{NS}$
$\mathcal{Q}$
$\mathcal{L}$
$x \in [X]$
$y \in [Y]$
P_x
M^y
$k \in [K]$
$p(k|x,y)$
$\mathcal{C}$
$\mathcal{Q}$
$\mathcal{NC}$
$S(\mathbf{p}) \leq S_{\mathcal{NC}}$
Contextuality
inequalities



Noncontextual set



Linear program [SSW18]

$$S(\mathbf{p}) = \ p(0|0,0) + p(0|1,1) + p(0|4,2) \leq 2.5$$

$$\frac{1}{2}(P_0 + P_1) \simeq \frac{1}{2}(P_2 + P_3) \simeq \frac{1}{2}(P_4 + P_5)$$

$$\frac{1}{3}([0|M_0] + [0|M_1] + [0|M_2]) \simeq \frac{1}{3}([1|M_0] + [1|M_1] + [1|M_2])$$

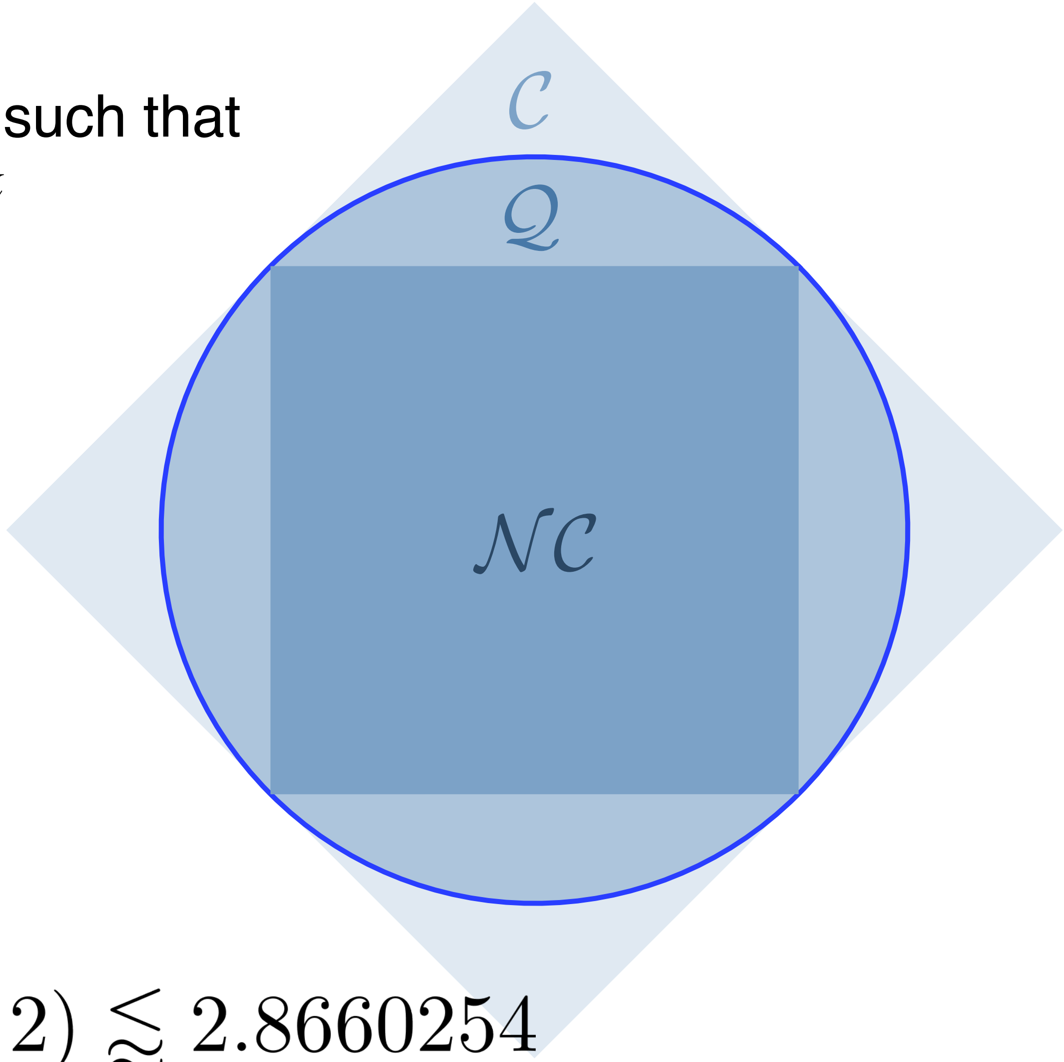
Quantum contextuality: when is $\mathbf{p} \in \mathcal{Q}$?

There exist density operators ρ_x and POVMs $\left\{E_k^y\right\}_k$ such that

$$\mathrm{Tr}(E_k^y \rho_x) = p(k|x,y)$$

$$\tfrac{1}{2}(\rho_0+\rho_1) = \tfrac{1}{2}(\rho_2+\rho_3) = \tfrac{1}{2}(\rho_4+\rho_5)$$

$$\tfrac{1}{3}(E_0^0+E_0^1+E_0^2) = \tfrac{1}{3}(E_1^0+E_1^1+E_1^2)$$



$$S(\mathbf{p}) = p(0|0,0) + p(0|1,1) + p(0|4,2) \lessapprox 2.8660254$$

Differences from nonlocality

pure states $|\Psi_x\rangle\langle\Psi_x|$

There exist ~~density operators~~ ρ_x and POVMs $\left\{E_k^y\right\}_k$ such that

$|\Psi_x\rangle\langle\Psi_x|$

$$\mathrm{Tr}(E_k^y \rho_x) = p(k|x,y)$$

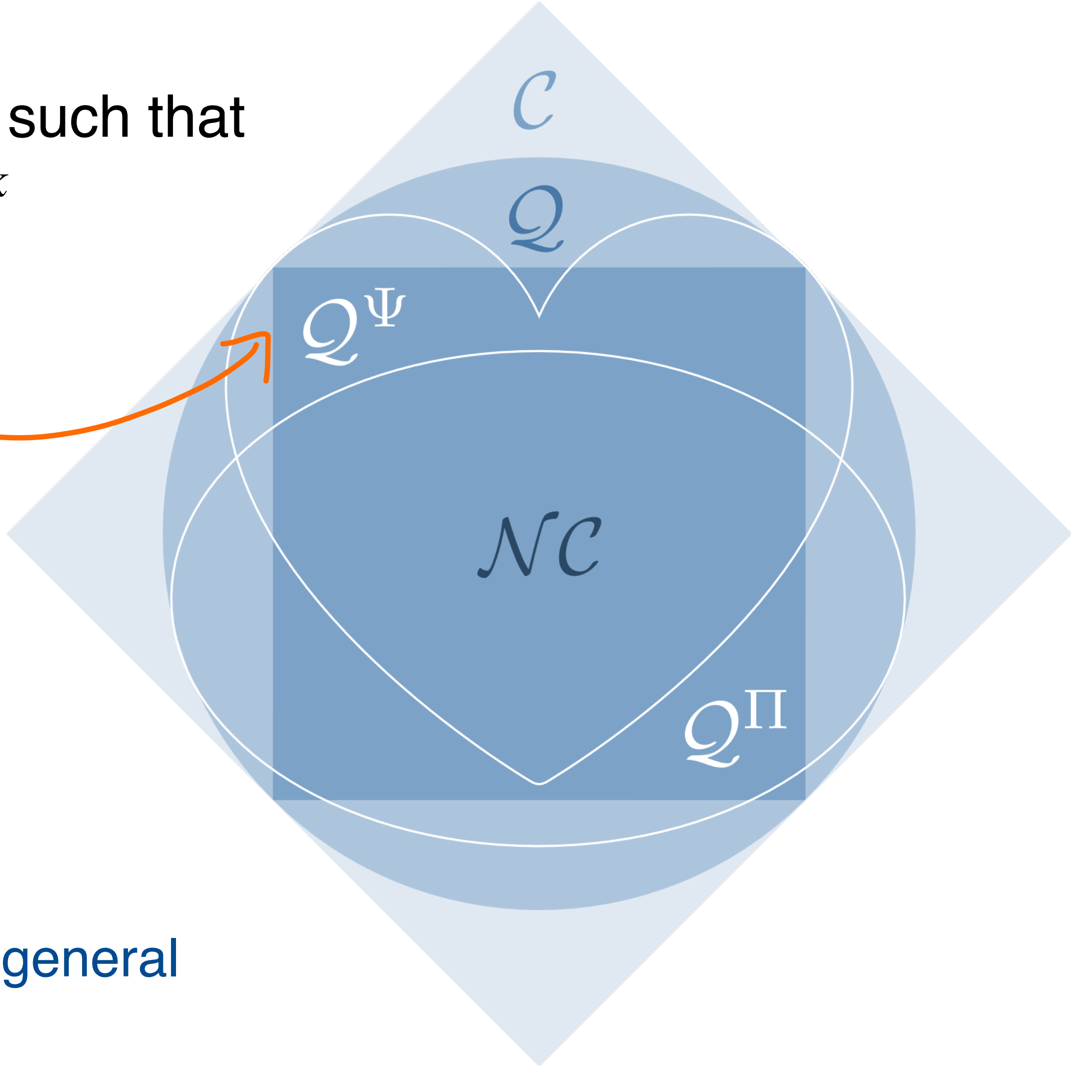
$$\tfrac{1}{2}(\rho_0+\rho_1) = \tfrac{1}{2}(\rho_2+\rho_3) = \tfrac{1}{2}(\rho_4+\rho_5)$$

$$\tfrac{1}{3}(E_0^0+E_0^1+E_0^2) = \tfrac{1}{3}(E_1^0+E_1^1+E_1^2)$$

non-trivial preparation equivalences $\quad \mathcal{Q}^\Psi \subsetneq \mathcal{Q}$

non-trivial measurement equivalences $\quad \mathcal{Q}^\Pi \subsetneq \mathcal{Q}$

in general



Differences from nonlocality

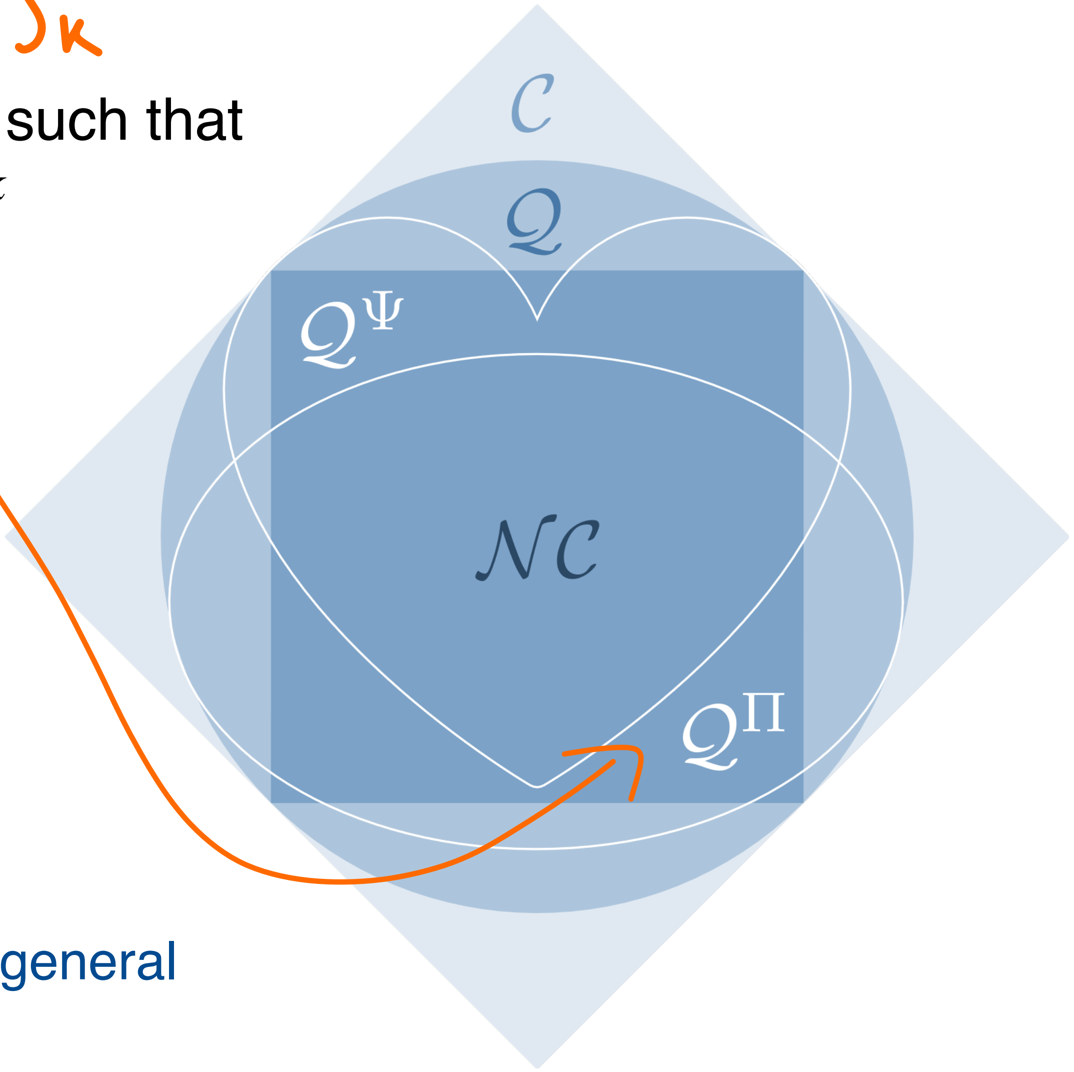
PVMs $\{P_k^y\}_k$

There exist density operators ρ_x and POVMs $\left\{E_k^y\right\}_k$ such that

$$\mathrm{Tr}(E_k^y \rho_x) = p(k|x,y)$$

$$\frac{1}{2}(\rho_0 + \rho_1) = \frac{1}{2}(\rho_2 + \rho_3) = \frac{1}{2}(\rho_4 + \rho_5)$$

$$\frac{1}{3}(E_0^0 + E_0^1 + E_0^2) = \frac{1}{3}(E_1^0 + E_1^1 + E_1^2)$$



non-trivial preparation equivalences $\quad \mathcal{Q}^{\Psi} \subsetneq \mathcal{Q}$

non-trivial measurement equivalences $\quad \mathcal{Q}^{\Pi} \subsetneq \mathcal{Q}$

in general

SDP relaxation

$p(k|x,y) = \mathrm{Tr}\left(E_k^y \rho_x\right)$ for some density operators ρ_x and POVMs $\left\{E_k^y\right\}_k$

Gram matrices

$(\Gamma_x)_{j,k} = \mathrm{Tr}(O_j^\dagger O_k \rho_x)$

$\{O_j \rho_x^{1/2}\}_j$

$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B)$

Hilbert-Schmidt

$$(\Gamma_x)_{j,k} = \mathrm{Tr}\left[(O_j \rho_x^{1/2})^\dagger O_k \rho_x^{1/2}\right]$$

$$= \mathrm{Tr}(\rho_x^{1/2} O_j^\dagger O_k \rho_x^{1/2})$$

$$= \mathrm{Tr}(O_j^\dagger O_k \rho_x)$$

SDP relaxation

$p(k|x,y) = \mathrm{Tr}\left(E_k^y \rho_x\right)$ for some density operators ρ_x and POVMs $\left\{E_k^y\right\}_k$

Gram matrices

$$(\Gamma_x)_{j,k} = \mathrm{Tr}(O_j^\dagger O_k \rho_x)$$

$\{O_1, O_2, \ldots\} = \{E_0^0, E_0^1, E_0^2\}$

	$\mathbb{I}$	E_0^0	E_0^1	E_0^2
$\mathbb{I}$	1	$\mathrm{Tr}(E_0^0\rho_x)$	$\mathrm{Tr}(E_0^1\rho_x)$	$\mathrm{Tr}(E_0^2\rho_x)$
E_0^0	$\mathrm{Tr}(E_0^0\rho_x)$	$\mathrm{Tr}({E_0^0}^2\rho_x)$		
E_0^1	$\mathrm{Tr}(E_0^1\rho_x)$	$\mathrm{Tr}(E_0^0 E_0^1\rho_x)$	$\mathrm{Tr}({E_0^1}^2\rho_x)$	
E_0^2	$\mathrm{Tr}(E_0^2\rho_x)$			$\mathrm{Tr}({E_0^2}^2\rho_x)$

$= \Gamma_x$

unbounded!

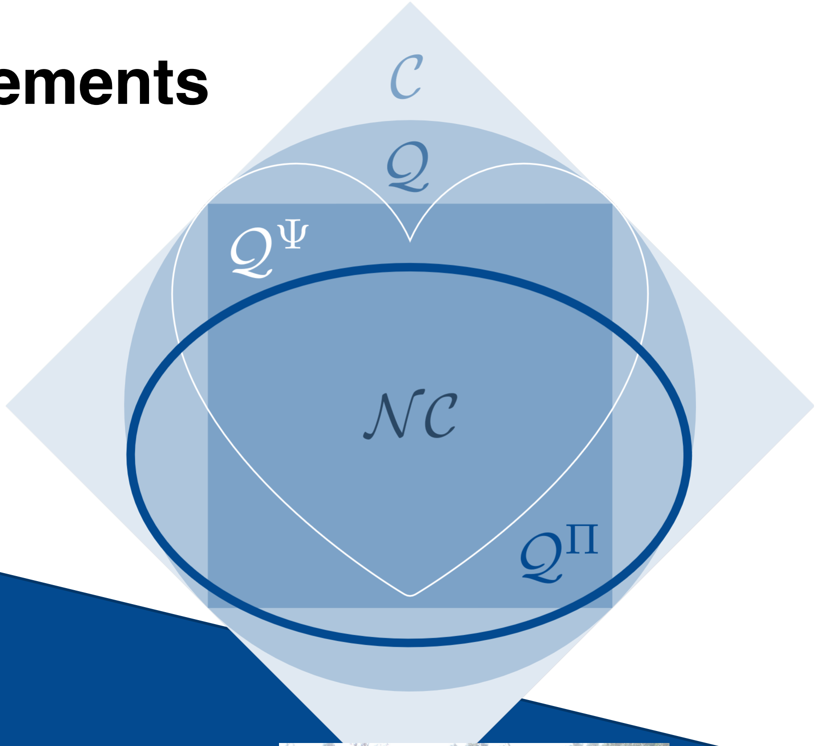
HOWEVER, currently very bad!

Relaxation:

operational equivalences

$$\exists \Gamma_x \geq 0 \quad \text{s.t.} \quad \frac{1}{2}\Gamma_0 + \frac{1}{2}\Gamma_1 = \frac{1}{2}\Gamma_2 + \frac{1}{2}\Gamma_3 = \frac{1}{2}\Gamma_4 + \frac{1}{2}\Gamma_5 \qquad (\Gamma_x)_{\mathbb{I},E_k^y} = p(k|x,y)$$

PROBLEM: cannot assume projective measurements



SOLUTION: a helpful lemma!

$$E = \frac{\mathbb{I}}{2} + \frac{U+U^\dagger}{4}$$

Instead use the operators U_k^y and $U_k^{y\,\dagger}$ to index the moment matrix.

$$(\Gamma_x)_{\mathbb{I},U_k^y} + (\Gamma_x)_{\mathbb{I},U_k^{y\,\dagger}} = 4\left(p(k|x,y) - \frac{1}{2}\right)$$

$$(\Gamma_x)_{U_k^y,U_k^y} = 1$$

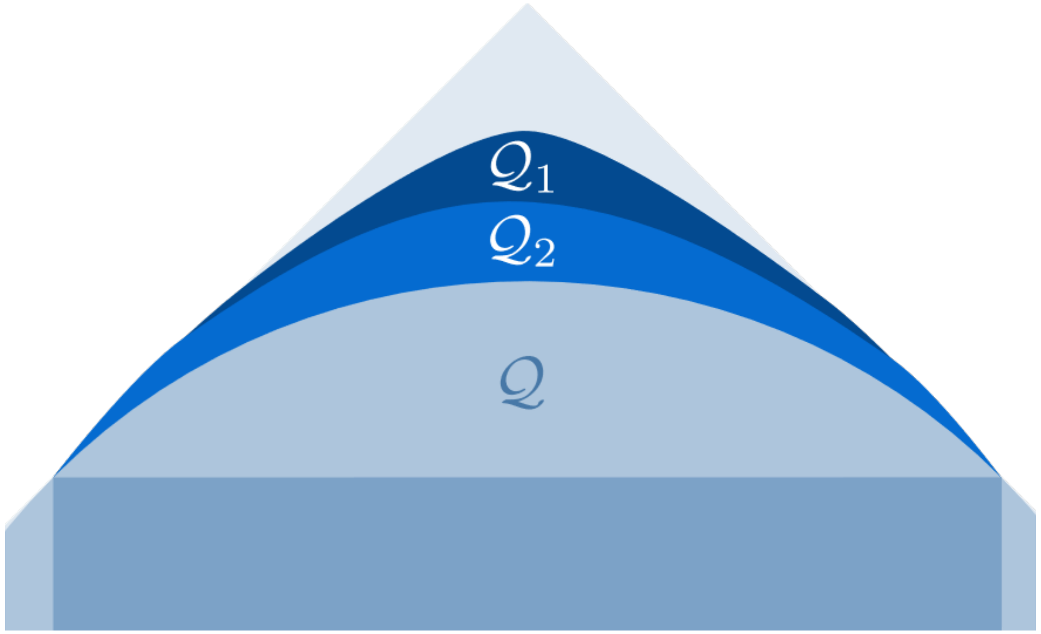
$p(k|x,y) = \mathrm{Tr}\left(E_k^y \rho_x\right)$ for some density operators ρ_x and POVMs $\left\{E_k^y\right\}_k$

LEVEL 1 Gram matrix of the vectors $\left\{ O\rho_x^{1/2} \mid O \in \{\mathbb{I}, U_k^y, (U_k^y)^\dagger\} \right\}$

LEVEL 2 Gram matrix of the vectors $\left\{ O_1 O_2 \rho_x^{1/2} \mid O_1, O_2 \in \{\mathbb{I}, U_k^y, (U_k^y)^\dagger\} \right\}$

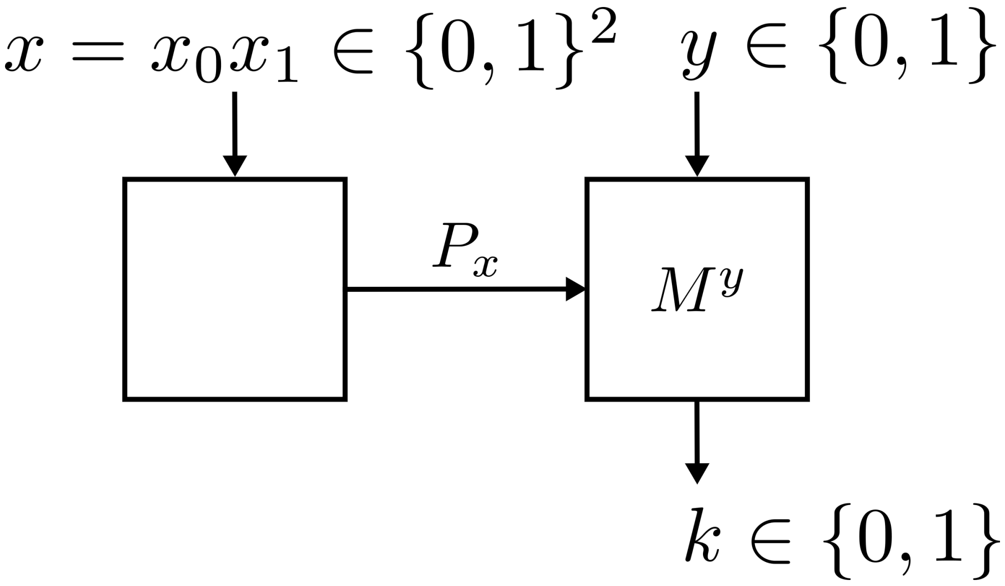
⋮

Contextual SDP hierarchy



[CFW21] and []

Parity-oblivious random access codes



$$\frac{1}{2}(P_{00} + P_{11}) \simeq \frac{1}{2}(P_{01} + P_{10})$$

Win when:

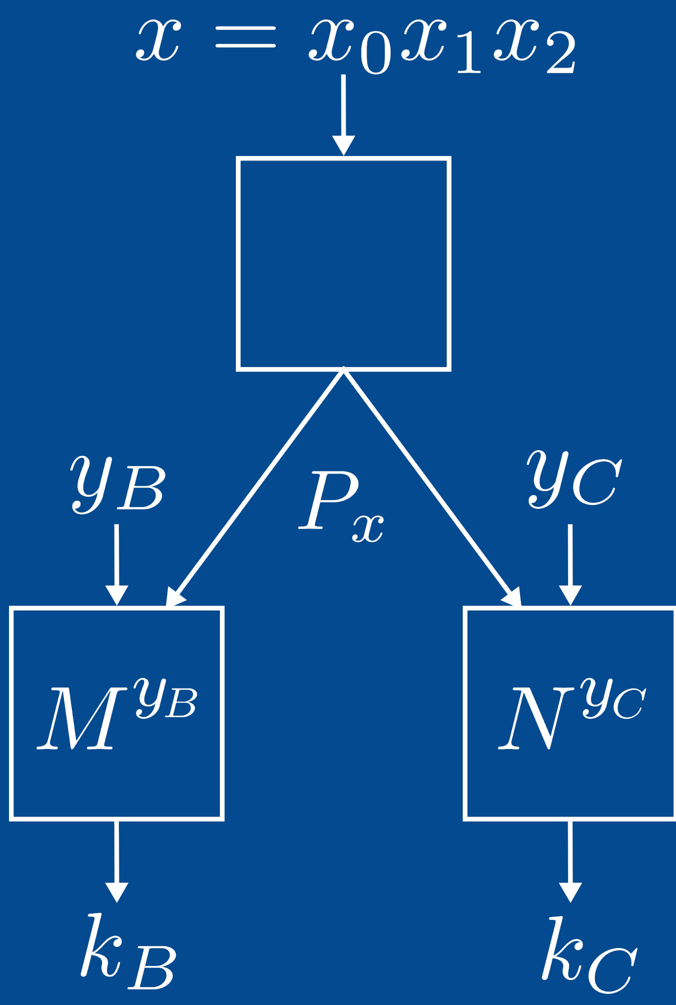
$$k = x_y$$

Average success probability:

$$S^{rac}(\mathbf{p}) = \frac{1}{8}\sum_{x,y} p(x_y|x,y)$$

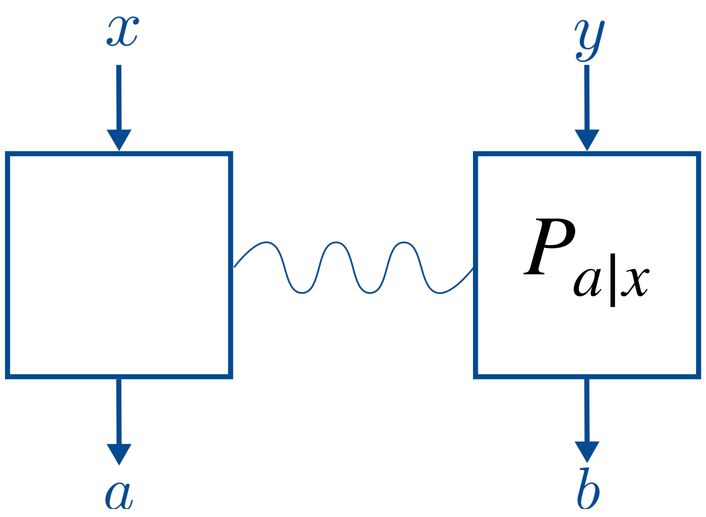
$$S^{rac}_{\mathcal{NC}} = \frac{3}{4} < S^{rac}_{\mathcal{Q}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

Monogamy of contextuality



$$S_B^{rac} + S_C^{rac} \leq 1.392 < 2S_Q^{rac}$$

Bell scenarios a subcase of contextuality scenarios?



No-signalling: $\sum_a p_X(a\,|\,0)P_{a|0} \simeq \sum_a p_X(a\,|\,1)P_{a|1} \simeq \ldots \simeq \sum_a p_X(a\,|\,1)P_{a|1}$

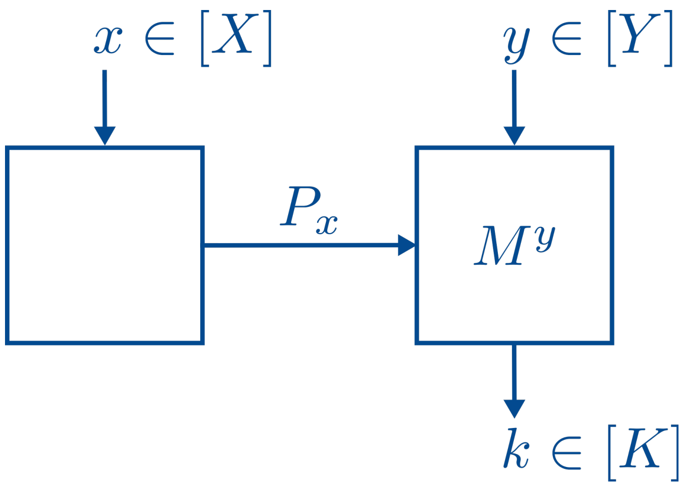
Quantum behaviours in Bell scenario

$\cong$

Quantum behaviours in the family of contextuality scenarios with these preparation equivalences

Bell scenarios a subcase of contextuality scenarios?

Kinda



If the contextuality scenario only has preparation equivalences and they are of the form

$$\sum_j \alpha_j P_j \simeq \sum_j \beta_j P_j \simeq \ldots \simeq \sum_j \zeta_j P_j$$

we can relabel these as

$$\sum_a p_X(a\,|\,0) P_{a|0} \simeq \sum_a p_X(a\,|\,1) P_{a|1} \simeq \ldots \simeq \sum_a p_X(a\,|\,n) P_{a|n}.$$

Quantum behaviours in contextuality scenario $\cong$ Subset of Bell scenario quantum behaviours with fixed marginals $p_X(a\,|\,x)$ for Alice

Are the hierarchies related?

If the contextuality scenario only has preparation equivalences and they are of the form

$$\sum_j \alpha_j P_j \simeq \sum_j \beta_j P_j \simeq \ldots \simeq \sum_j \zeta_j P_j$$

we can use NPA!

Are the hierarchies are equivalent in this case? No counter example found.

Some other things we don't know

- Does the contextuality hierarchy converge?
- How fast does either hierarchy converge?

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