Gleason's Theorem and its extensions

Gleason's theorem is a result about Hilbert space, and can be applied to quantum theory to give various fundamental insights. We will motivate the relevant definitions from the following perspective

AXIOMATISING QUANTUM THEORY

QT often presented via postulates, e.g.,

$\rho : \mathcal{H} \to \mathcal{H}$
- linear
- positive $\langle \psi, \rho\psi \rangle \geq 0 \;\; \forall \psi \in \mathcal{H}$
- unit trace $\sum_j \langle \psi_j, \rho \psi_j \rangle = 1$ for an orthonormal basis $\{\psi_j\}$

SOME POSTULATES FOR QT

(H) Every quantum system has an associated separable Hilbert space, $\mathcal{H}$

(S) The state of the quantum system can be described by a density operator, ρ, on $\mathcal{H}$

(M) A (discrete) measurement on the system can be described by a sequence $(\Pi_1, \Pi_2, \dots)$ s.t. $\sum_j \Pi_j = I_{\mathcal{H}}$ (identity operator) of mutually orthogonal self-adjoint projections on $\mathcal{H}$ (each projection describing a disjoint outcome of the measurement)

(B) The probability of observing outcome Π_j when the system being measured resides in state ρ is given by $\mathrm{Tr}(\Pi_j \rho)$

+ dynamics and composition

$\Pi_j : \mathcal{H} \to \mathcal{H}$
- linear
- $\Pi_j = (\Pi_j)^2$
- $\Pi_j^{\dagger} = \Pi_j$

$\mathcal{P}(\mathcal{H})$ = self adjoint projections

- $\Pi_j \Pi_k = 0$ if $j \neq k$ ← mutually orthogonal

These postulates are quite abstract. Various efforts have been made to find alternative axioms for quantum theory. Gleason's theorem was an important early step in this direction.

Question: given (M) what states are possible?

What is the role of a state in a theory?

To tell you the probabilities of the outcomes of performing an experiment

⇓

A state should be a probability assignment on all the possible outcomes of all the possible measurements.

Remark: we assume in (M) that if two measurement outcomes are represented by the same projections they are "the same" i.e. operationally indistinguishable

(M) ⇒ for every state there exists a probability measure μ on $\mathcal{P}(\mathcal{H})$

$$\mu : \mathcal{P}(\mathcal{H}) \to [0,1] \text{ s.t.:}$$

$$\mu(\Pi_1) + \mu(\Pi_2) + \dots = 1 \qquad (*)$$

sum because outcomes disjoint

for all sequences $(\Pi_1, \Pi_2, \dots)$ of mutually orthogonal projections such that $\sum_j \Pi_j = I_{\mathcal{H}}$

GLEASON'S THEOREM

A. M. Gleason, J. Math. Mech. **6** (1957) 885

If $\dim(\mathcal{H}) \geq 3$ for every probability measure μ on $\mathcal{P}(\mathcal{H})$ $\exists$ a density operator ρ on $\mathcal{H}$

$$\mu(\Pi) = \mathrm{Tr}(\Pi\rho) \qquad \forall\; \Pi \in \mathcal{P}(\mathcal{H})$$

Answer: density operators! (in dim ≥ 3)

Remark:

Another way to express (*) is $\underbrace{\mu(\Pi_1) + \mu(\Pi_2) + \dots = \mu(\Pi_1 + \Pi_2 + \dots)}_{\text{countably additive}}$ and $\mu(I_{\mathcal{H}}) = 1$

Gleason's theorem tells us any countably additive functional on $\mathcal{P}(\mathcal{H})$ extends to a unique linear (countable additivity + homogeneity $f(aA) = af(A)$) functional on self-adjoint operators on $\mathcal{H}$

Later it was shown that countable additivity can be relaxed to finite additivity $\mu(\Pi_1) + \mu(\Pi_2) = \mu(\Pi_1 + \Pi_2)$ whilst preserving Gleason's result

E. Christensen, Commun. Math. Phys. **86** (1982) 529

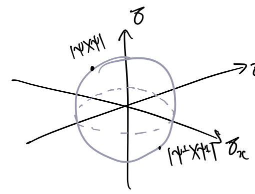
Under the assumption (M) there are no additional possible states under our definition to those in standard quantum theory. The postulates (S) and (B) can be replaced by our definition of a state.

UNLESS $\dim(\mathcal{H}) = 2$

What goes wrong? ← rank-one projections ⟶ elements of Bloch sphere

$\mathcal{P}(\mathbb{C}^2) = \{ I_{\mathbb{C}^2}, 0, |\psi\rangle\langle\psi| \}$

Measurements: $\{I_{\mathbb{C}^2}\}$, $\{ |\psi\rangle\langle\psi|, |\psi^\perp\rangle\langle\psi^\perp| \}$



Each projection only appears in one equality $\mu(|\psi\rangle\langle\psi|) + \mu(|\psi^\perp\rangle\langle\psi^\perp|) = 1$.
The values of μ on non-orthogonal projections are not related.

$\exists$ many measures μ that have no linear extension i.e. $\neq \mathrm{Tr}(\Pi\rho)$ for some ρ

e.g. $\mu(|\psi\rangle\langle\psi|) = \begin{cases} 1 & \text{for } |\psi\rangle \text{ on the northen hemisphere of Bloch sphere} \\ 0 & \text{for } |\psi\rangle \text{ '' southern ''} \end{cases}$

or $ = \begin{cases} 1 & \text{for } |\psi\rangle = |0\rangle \\ 0 & \text{for } |\psi\rangle = |1\rangle \\ \tfrac{1}{2} & \text{otherwise} \end{cases}$

So we can't entirely remove the (S) and (B) postulates due to two-level quantum systems :(

EXTENSION 1:
BUT WHAT ABOUT POVMs?

When we consider generalised measurements our postulate (M) changes:

$\mathcal{E}(\mathcal{H}) = \{ E \in \mathcal{L}(\mathcal{H}) \mid 0 \leq E \leq I_{\mathcal{H}} \}$

(GM) A discrete measurement on the system can be described by a sequence $(E_1, E_2, \ldots)$ of effects on $\mathcal{H}$ s.t. $\sum_j E_j = I_{\mathcal{H}}$
(each effect describing a disjoint outcome of the measurement)

Using the same reasoning as before we find

(GM) $\Rightarrow$ for every state $\exists$ a generalised probability measure μ on $\mathcal{E}(\mathcal{H})$

$$\mu : \mathcal{E}(\mathcal{H}) \to [0,1] \quad \text{s.t.}$$
$$\mu(E_1) + \mu(E_2) + \ldots = 1$$
$\forall$ sequences $(E_1, E_2, \ldots)$ of effects such that $\sum_j E_j = I_{\mathcal{H}}$

BUSCH'S GLEASON-TYPE THEOREM

P. Busch, Phys. Rev. Lett. **91** (2003) 120403

For any separable Hilbert space $\mathcal{H}$ and generalised probability measure $\mu : \mathcal{E}(\mathcal{H}) \to [0,1]$ $\exists$ a density operator ρ on $\mathcal{H}$ s.t.

$$\mu(E) = \mathrm{Tr}(E\rho) \quad \forall \; E \in \mathcal{E}(\mathcal{H})$$

Holds for all separable Hilbert spaces! YAY! :)

So (H), (GM) + definition of a state can replace (H), (S), (GM), (B)

Now we know if we can motivate the formulation of generalised measurements, states and the Born rule follow easily.

EXTENSION 1a
DO WE NEED ALL POVMs?

We now consider finite dimensional $\mathcal{H} \cong \mathbb{C}^d$. In this case POVMs $(E_1, \ldots, E_n)$ with a finite number of outcomes are known to be sufficient, i.e., $\mu : \mathcal{E}(\mathcal{H}) \to [0,1]$, s.t. $\mu(E_1) + \mu(E_2) + \ldots + \mu(E_n) = 1 \;\; \forall$ POVMs $(E_1, \ldots, E_n)$ $\Rightarrow$ $\mu(E) = \mathrm{Tr}(E\rho) \;\; \forall \; E \in \mathcal{E}(\mathcal{H})$.

Note that all $E \in \mathcal{E}(\mathcal{H})$ appear in finite POVMs.

Furthermore, we can reduce to three or fewer outcome POVMs (E_1, E_2, E_3).

For $E_1, E_2 \in \mathcal{E}(\mathcal{H})$ s.t. $E_1 + E_2 \in \mathcal{E}(\mathcal{H})$, let $E_3 = I_{\mathcal{H}} - E_1 - E_2$

We have

$\mu(E_1) + \mu(E_2) + \mu(E_3) = 1$

and $\Rightarrow \mu(E_1) + \mu(E_2) = \mu(E_1 + E_2)$ $\overset{\text{(by induction)}}{\Rightarrow}$ $\mu(E_1) + \mu(E_2) + \ldots + \mu(E_n) = 1 \;\; \forall$ POVMs $(E_1, \ldots, E_n)$

$\mu(E_1 + E_2) + \mu(E_3) = 1$

Here again we have finite additivity so Busch's result shows that finite additivity on effects $\Rightarrow$ a linear extension to the full vector space of self-adjoint operators. The analogue of this problem on the reals is the subject of Cauchy's functional equation:

$$f(x)+f(y) = f(x+y) \quad \text{for } f:\mathbb{R}\to\mathbb{R} \text{ and all } x,y\in\mathbb{R}.$$

$\exists$ non-linear f satisfying this equation but they are very unusual and ruled out by minimal extra requirements, e.g. Lesbesgue measurability, continuity at a single point, or being bounded on any set of positive measure. We look at the connection with Busch's theorem here. VJW, S. Weigert Found. Phys. 49 (2019) 594

We can also reduce to projective-simulable POVMs.

M Oszmaniec, L Guerini, P Wittek, A Acín
Physical review letters 119 (19), 190501

E.g. imagine you perform either $(\underset{\text{outcome } 0}{|0\rangle\langle 0|}, \underset{\text{outcome } 1}{|1\rangle\langle 1|})$ or $(\underset{\text{outcome } 0}{|+\rangle\langle +|}, \underset{\text{outcome } 1}{|-\rangle\langle -|})$ with equal probability

probability you see outcome 0: $\frac{1}{2}\mathrm{Tr}(|0\rangle\langle 0|\rho) + \frac{1}{2}\mathrm{Tr}(|+\rangle\langle +|\rho) = \mathrm{Tr}[(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|)\rho]$

" 1: $\frac{1}{2}\mathrm{Tr}(|1\rangle\langle 1|\rho) + \frac{1}{2}\mathrm{Tr}(|-\rangle\langle -|\rho) = \mathrm{Tr}[(\frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|-\rangle\langle -|)\rho]$

} implemented POVM $\left(\frac{1}{2}(|0\rangle\langle 0| + |+\rangle\langle +|), \frac{1}{2}(|1\rangle\langle 1| + |-\rangle\langle -|)\right)$

Finally, we can reduce to only considering two-outcome (projective simulable) POVMs and three outcome projective simulable POVMs of the forms

$$\mathbb{T}_e = \left[\!\left[\frac{e}{2},\frac{e}{2},I-e\right]\!\right] \quad \textit{and} \quad \mathbb{T}_{e,e'} = \left[\!\left[\frac{e}{2},\frac{e'}{2},I-\frac{(e+e')}{2}\right]\!\right] \qquad \text{for any } e,e'\in\mathcal{E}(\mathcal{H}) \text{ s.t. } e+e'\in\mathcal{E}(\mathcal{H})$$

VJW and S Weigert *J. Phys. A*, 52:055301, 2019.

EXTENSION 2
UNENTANGLED PROJECTIVE MMTs

Nolan R Wallach. An unentangled Gleason's theorem. *Contemp Math*, 305:291–298, 2002. doi: 10.1090/conm/305/05226.

Consider a composite Hilbert space $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$. Wallach showed if each $\mathcal{H}_j$ has finite dimension at least three then Gleason's result can be proven using only unentangled projective measurements, i.e.,

$$(\Pi_1, \Pi_2, \ldots, \Pi_m)$$

such that each Π_j projects onto a subspace of $\mathcal{H}$ which has a basis of product vectors (wrt the 'partition' $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$).

E.g. any rank-one projection $|\psi\rangle\langle\psi|$ such that ψ is a product vector $\psi = \psi_1\otimes\psi_2\otimes\cdots\otimes\psi_n$, or $|00\rangle\langle 00| + |11\rangle\langle 11|$ $\leftarrow$ on $\mathbb{C}^2\otimes\mathbb{C}^2$, note that even though this subspace admits an entangled basis $\{\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)\}$ it also admits an unentangle basis $\{|00\rangle, |11\rangle\}$ so is unentangled.

EXTENSION 3
OTHER VON NEUMANN ALGEBRAS

E. Christensen, Commun. Math. Phys. **86** (1982) 529

The bounded linear operators $B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ are an example of a vN algebra. Specifically, they are a type I_d factor, where d is the (possibly infinite) dimension of $\mathcal{H}$. Any vN algebra can be decomposed into a direct integral of factors. As long as none of these factors are type I_2 $(B(\mathbb{C}^2))$ the projections of the vN algebra admit a Gleason theorem.

EXTENSION 4
GENERAL PROBABILISTIC THEORIES

VJW, and Stefan Weigert. "General probabilistic theories with a Gleason-type theorem." *Quantum* 5 (2021): 588.

F. J. Yeadon, *Measures on projections in W*-algebras of type II_1*, Bulletin of the London Mathematical Society, 15 (1983), pp. 139–145.

If we had found a different GPT that described nature, could we have proved a Gleason-type theorem?

RECAP: <u>measurements in GPTs</u>.

Each system has an effect space $\mathcal{E}$, which is a convex subset of a real vector space, V, and contains a special vector $\underline{u}$, the unit effect. Possible outcomes of measurements are described by effects $\underline{e}\in\mathcal{E}$. An outcome that always occurs in any state is described by the unit effect $\underline{u}$.

A measurement is the described by a sequence of effects $(\underline{e}_1, \underline{e}_2, \ldots)$ such that $\underline{e}_1 + \underline{e}_2 + \ldots = \underline{u}$.

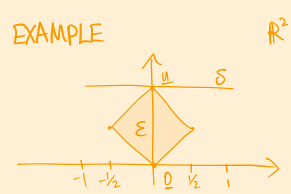
For every $\underline{e}\in\mathcal{E}$, $\underline{u}-\underline{e}\in\mathcal{E}$, so the two outcome measurement $(\underline{e}, \underline{u}-\underline{e})$ is in the theory.

<u>States in GPTs</u>

Each system has a state space $\mathcal{S}$, which is a convex subset of the real vector space V.

The probability of $\underline{e}$ when the system is in state $\underline{\omega}$ is $\underline{e}\cdot\underline{\omega}$.

EXAMPLE $\mathbb{R}^2$



The largest possible state space allowed by the GPT framework if the system has effect space $\mathcal{E}$ is

$\{\underline{w} \in V \mid 0 \le \underline{e} \cdot \underline{w} \le 1, \forall\, \underline{e} \in \mathcal{E} \text{ and } \underline{u} \cdot \underline{w} = 1\}$ (**)

We can now define a generalised probability measure analogously to the quantum case as a map $\mu: \mathcal{E} \to [0,1]$ s.t.

$\mu(\underline{e}_1) + \mu(\underline{e}_2) + \ldots = 1$ for any sequence $(e_1, e_2, \ldots)$ of effects describing a mmt in the theory.

We find that $\mu(\underline{e}) = \underline{e} \cdot \underline{w}$ for some vector $\underline{w}$ in the unrestricted state space

In quantum theory every generalised probability measure has a corresponding state, i.e. density operator. If we say that a GPT has a Gleason-type theorem when every generalised probability measure on the effect space (of each system in the theory) has a corresponding state, then having a Gleason-type theorem is equivalent to all state spaces being unrestricted, i.e. given by (**). This is a weaker requirement than the standard no-restriction hypothesis.

effect space is as large as possible given the state space

$\mathcal{E} = \{\underline{e} \in V \mid 0 \le \underline{e} \cdot \underline{w} \le 1 \ \forall\, \underline{w} \in \mathcal{S}\}$

GPTs with a Gleason-type theorem can be characterised as noisy unrestricted GPTs, which relates to the no-restriction hypothesis.

OPEN QUESTION: which GPTs admit an analogue of Gleason's original result, where the probability measures on the analogue of projective mmts are all states in the theory?

GLEASON'S THEOREM $\Rightarrow$ THE KOCHEN-SPECKER THEOREM

KS theorem: there is no deterministic, non-contextual ontological model for quantum theory

In such a model an ontic state would need to select one outcome from every projective measurement $(\Pi_1, \Pi_2, \ldots)$ such that if Π is selected in one measurement it is also selected in any other measurement it appears in, e.g.

$(|0\rangle\langle 0|, |1\rangle\langle 1|, |2\rangle\langle 2|)$ $(|+\rangle\langle +|, |-\rangle\langle -|, |2\rangle\langle 2|)$

If ontic state chooses this outcome, it must also choose this outcome.

Such an ontic state can therefore be identified with a map, $\lambda: \mathcal{P}(\mathcal{H}) \to \{0,1\}$ s.t. for any sequence $(\Pi_1, \Pi_2, \ldots)$ of mutally orthogonal projections from $\mathcal{P}(\mathcal{H})$, $\lambda(\Pi) = 1$ for exactly one $\Pi \in (\Pi_1, \Pi_2, \ldots)$

It follows that λ is a probability measure!

However, Gleason's theorem tells us that any probability measure can be written as $\mu(\Pi) = \mathrm{Tr}(\Pi \rho)$ for some density operator ρ. Such a map is one on at most one rank-one projection so is not an ontic state.